EXAM 1

Math 216, 2021 Spring.

Name: Solutions NetID: Student ID:

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

It is strongly advised that you use black pen only, since that will be most clear in scanning your work.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature:

1. *(18 pts)* The matrix A is

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 4 \\ 2 & 6 & 4 \end{pmatrix}$$

(a) Find a matrix E for which EA is the reduced row echelon form R for A.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 2 & 6 & 4 & 0 & 0 & 1 \end{array}\right)$$

(b/c $EA = R$ $\Rightarrow$ $E(A|I) = (R|E)$)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 2 & 4 & -2 & 0 & 1 \end{array}\right)\begin{array}{l} 1 \\ 2 \\ 3-2\cdot 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right)\begin{array}{l} 1-2 \\ 2/2 \\ 3-2 \end{array}$$

$$\underbrace{}_{R} \quad \underbrace{}_{E}$$

(b) Find the complete set of solutions to the system $A\vec{x} = (3, 2, 8)$.

$$A\vec{x} = \vec{b} \xLeftrightarrow{E\cdot} R\vec{x} = E\vec{b} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -4 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{array}{l} x_1 = 1 + 4x_3 \\ x_2 = 1 - 2x_3 \end{array} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 + 4x_3 \\ 1 - 2x_3 \\ x_3 \end{pmatrix}$$

(c) We are interested in finding a vector $\vec{c}$ for which $A\vec{x} = E^{-1}\vec{c}$ has no solutions. Explain a simple strategy (NOT guess-and-check!) that would lead to finding such a vector.

$A\vec{x} = E^{-1}\vec{c} \Leftrightarrow R\vec{x} = \vec{c}$, so we can make such a vector $\vec{c}$ just by putting a nonzero value in the third coordinate – e.g., $\vec{c} = (0,0,1)$.

2. *(15 pts)* The matrix A is the product BC, where B is 3×5 and C is 5×3. B is known to have two identical rows. Show that A is not invertible. (Explain your reasoning with all details!)

Two identical rows of B, used as coefficients in linear combinations of rows of C, will make two identical rows of A.

Thus rref(A) will have a row of zeroes, and so A is not invertible.

(NB — we can <u>not</u> argue that $\det B = 0$ and use multiplicativity to conclude $\det A = 0$, because B is not square so $\det B$ is not defined!)

3. *(18 pts)* Use a row reduction to find the inverse of A, the determinant of A, and to write A as a product of elementary matrices.

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 5 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 \end{array}\right)\begin{matrix} 3 \\ 2 \\ 1 \end{matrix} \leftarrow \det = -\det A \qquad E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & -9 & 1 & -2 & 0 \end{array}\right)\begin{matrix} 1 \\ 2 \\ 3 - 2\cdot 2 \end{matrix} \leftarrow \det = -\det A \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/9 & 2/9 & 0 \end{array}\right)\begin{matrix} 1 \\ 2 \\ 3/-9 \end{matrix} \leftarrow \det = \frac{1}{9}\det A \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/9 & -2/9 & 1 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/9 & 2/9 & 0 \end{array}\right)\begin{matrix} 1 - 3 \\ 2 \\ 3 \end{matrix} \leftarrow \det = \frac{1}{9}\det A \qquad E_4 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Big(\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{R=I,\text{ so}}\Big|\underbrace{\begin{array}{ccc} 1/9 & -2/9 & 1 \\ 5/9 & -1/9 & 0 \\ -1/9 & 2/9 & 0 \end{array}}_{\text{this is } A^{-1}.}\Big)\begin{matrix} 1 \\ 2 - 5\cdot 3 \\ 3 \end{matrix} \leftarrow \det = \frac{1}{9}\det A = 1 \qquad E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \det A = 9$$

$$E_5 E_4 E_3 E_2 E_1 A = R = I \implies A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1}$$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -9 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

4. *(15 pts)* Find a basis for the plane $x + 2y + 3z = 0$ thought of as a subspace of $\mathbb{R}^3$, and show that it is a basis without making any use of dimension.

The vectors $\vec{v} = (2,-1,0)$ and $\vec{w} = (3,0,-1)$ are both in the plane P.

Neither is a multiple of the other, so $\{\vec{v}, \vec{w}\}$ is linearly independent.

To show they span P, we consider

$$c_1 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2b_2 - 3b_3 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\left.\begin{aligned} 2c_1 + 3c_2 &= -2b_2 - 3b_3 \\ -1\,c_1 \quad &= b_2 \\ -1\,c_3 &= b_3 \end{aligned}\right\} \Rightarrow c_1 = -b_2,\ c_2 = -b_3$$

$$\Rightarrow 2c_1 + 3c_2 = -2b_2 - 3b_3$$

So there is a solution for every $\vec{b}$, and thus $\{\vec{v}, \vec{w}\}$ spans P.

So $\{\vec{v}, \vec{w}\}$ is a basis for P.

5. *(18 pts)*

(a) Describe what it would mean for the list of functions below to be linearly dependent, using an equation explicitly involving those functions and a description of what must be true about that equation.

$$\{\underbrace{4\sin(x-\pi/5)}_{f_1}, \underbrace{2\cos(x-\pi/7)}_{f_2}, \underbrace{3\sin(x-\pi/11)}_{f_3}\}$$

We need

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$$

to have a significant solution.

(b) The general fact that $[c_1v_1 + \ldots + c_nv_n]_\beta = c_1[v_1]_\beta + \ldots + c_n[v_n]_\beta$ allows for rewriting the equation above as equivalent to an equation involving m vectors in $\mathbb{R}^k$. What are these values of m and k, and an example of such a basis β?

The angle addition formulas for sine and cosine show that $f_1, f_2, f_3 \in V$ with basis $\beta = \{\sin x, \cos x\}$.

With this basis, taking coordinates gives us

$$c_1\begin{pmatrix}\cdot\\ \cdot\end{pmatrix} + c_2\begin{pmatrix}\cdot\\ \cdot\end{pmatrix} + c_3\begin{pmatrix}\cdot\\ \cdot\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{so } m=3,\ k=2.$$

(c) Give a rank argument to conclude whether the original list is linearly independent or linearly dependent.

The above vector equation is equivalent to an augmented matrix with shape

$$\left(\begin{array}{ccc|c} \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \end{array}\right)$$

There are only two rows so rank ≤ 2, and there are 3 columns, so there must be a free variable.

$\Rightarrow \infty$ solutions $\Rightarrow$ significant solutions $\Rightarrow$ lin. dep.

6. *(16 pts)*

(a) Find the Wronskian of $\{3\sin^2 x, 4\cos^2 x, 2\}$ by direct computation from the definition. What can you conclude ONLY from the result of this computation? Is there other information about these functions individually that combined with the result of the above computation allows you to conclude more?

$$w(x) = \det\begin{pmatrix} 3\sin^2 x & 4\cos^2 x & 2 \\ 6\sin x\cos x & -8\sin x\cos x & 0 \\ 12(\cos^2 x-\sin^2 x) & -16(\cos^2 x-\sin^2 x) & 0 \end{pmatrix}$$

$$= 2\Big(\big(-96(\sin x\cos x)(\cos^2 x-\sin^2 x)\big) - \big(-96(\sin x\cos x)(\cos^2 x-\sin^2 x)\big)\Big) = 0.$$

Using only this result we conclude nothing.

With the additional fact that these functions are analytic, we conclude that the list is linearly dependent.

(b) Bob claims to have computed the Wronskian of $\{\sin^2 x, \cos^2 x, \tan^2 x, \sec^2 x\}$, and that his result is

$$w(x) = \sin^2 x\cos^4 x + \tan^2 x\sec^4 x$$

Is he right? Explain fully.

Pythagorean trig identities give us

$$1(\sin^2 x) + 1(\cos^2 x) + 1(\tan^2 x) - 1(\sec^2 x) = 0$$

So the list is linearly dependent, and thus the actual Wronskian must be $w(x) = 0$.

So Bob is wrong.