EXAM 3

Math 216, 2020 Fall.

Name: Solutions NetID:________ Student ID:________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

It is strongly advised that you use black pen only, since that will be most clear in scanning your work.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

(Scratch space. Nothing on this page will be graded!)

1. *(20 pts)*

(a) Find the eigenvalues and eigenvectors of the matrix A below.

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & -1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{pmatrix}$$

$$= 1\big(1-(3-\lambda)\big) + (2-\lambda)\big(\lambda^2-5\lambda+6-(-1)\big)$$

$$= -\lambda^3 + 7\lambda^2 - 16\lambda + 12$$

$p(2) = 0 \Rightarrow (\lambda - 2)$ is a factor

$$\begin{array}{r|l} & -\lambda^2 + 5\lambda - 6 \\ \hline \lambda - 2 & -\lambda^3 + 7\lambda^2 - 16\lambda + 12 \\ & \underline{-\lambda^3 + 2\lambda^2} \\ & \quad 5\lambda^2 - 16\lambda + 12 \\ & \quad \underline{5\lambda^2 - 10\lambda} \\ & \quad\quad -6\lambda + 12 \\ & \quad\quad \underline{-6\lambda + 12} \\ & \quad\quad\quad 0 \end{array}$$

And

$$-\lambda^2 + 5\lambda - 6 = -(\lambda-2)(\lambda-3)$$

So

$$p(\lambda) = -(\lambda-2)^2(\lambda-3)$$

Eigenvalues are $2, 3$.

<u>For $\lambda = 2$:</u>

$$A - \lambda I = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

rows not parallel, so
$\dim(RS) = 2 = \text{rank}$
$\Rightarrow \dim(NS) = 3 - 2 = 1$

So $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \in NS(A - \lambda I)$ is the only eigenvector.

(extra space for questions from other side)

For $\lambda = 3$:

$$A - \lambda I = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

rows not parallel, so $\dim(RS) = 2 = \text{rank}$

$\Rightarrow \dim(NS) = 3 - 2 = 1$

So $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in NS(A - \lambda I)$ is the only eigenvector.

(b) List all of the Jordan forms that are possible given the results of part (a).

The eigenvalue block structure ($\lambda_1 = 2$ $m_1 = 2$, $\lambda_2 = 3$ $m_2 = 1$) allows two possible Jordan forms:

$$J_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad J_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

But J_2 would require two eigenvectors for $\lambda = 2$ and A has only 1, so it cannot be the Jordan form.

So J_1 is the only possibility.

2. *(20 pts)* The inner product space V consists of the vector space $\mathbb{R}^4$, but using the inner product $\langle \vec{v}, \vec{w} \rangle = B\vec{v} \cdot B\vec{w}$ with

$$B = \begin{pmatrix} 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

(a) Confirm that the first three vectors listed in the basis α (below) for V below are orthonormal.

$$\alpha = \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}}_{\vec{v}_2}, \underbrace{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}}_{\vec{v}_3}, \underbrace{\begin{pmatrix} -1 \\ 2 \\ 2 \\ -2 \end{pmatrix}}_{\vec{v}_4} \right\}$$

$$\left.\begin{aligned}
\|\vec{v}_1\|^2 &= \langle \vec{v}_1, \vec{v}_1 \rangle = (B\vec{v}_1)\cdot(B\vec{v}_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \|\vec{v}_1\| = 1 \\
\|\vec{v}_2\|^2 &= \langle \vec{v}_2, \vec{v}_2 \rangle = (B\vec{v}_2)\cdot(B\vec{v}_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \|\vec{v}_2\| = 1 \\
\|\vec{v}_3\|^2 &= \langle \vec{v}_3, \vec{v}_3 \rangle = (B\vec{v}_3)\cdot(B\vec{v}_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \|\vec{v}_3\| = 1
\end{aligned}\right\} \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \text{ are normalized.}$$

$$\left.\begin{aligned}
\langle \vec{v}_1, \vec{v}_2 \rangle &= (B\vec{v}_1)\cdot(B\vec{v}_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0 \\
\langle \vec{v}_1, \vec{v}_3 \rangle &= (B\vec{v}_1)\cdot(B\vec{v}_3) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0 \\
\langle \vec{v}_2, \vec{v}_3 \rangle &= (B\vec{v}_2)\cdot(B\vec{v}_3) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0
\end{aligned}\right\} \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \text{ are orthogonal.}$$

(b) Perform Gram-Schmidt orthonormalization on the basis α to find an orthonormal basis for V.

$$\vec{X}_4 = \vec{V}_4 - \langle \vec{V}_4, \vec{V}_1 \rangle \vec{V}_1 - \langle \vec{V}_4, \vec{V}_2 \rangle \vec{V}_2 - \langle \vec{V}_4, \vec{V}_3 \rangle \vec{V}_3$$

$$\langle \vec{V}_4, \vec{V}_1 \rangle = (B\vec{v}_4)\cdot(B\vec{v}_1) = \begin{pmatrix}1\\2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 1$$

$$\langle \vec{V}_4, \vec{V}_2 \rangle = (B\vec{v}_4)\cdot(B\vec{v}_2) = \begin{pmatrix}1\\2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = 2$$

$$\langle \vec{V}_4, \vec{V}_3 \rangle = (B\vec{v}_4)\cdot(B\vec{v}_3) = \begin{pmatrix}1\\2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = 0$$

$$\vec{X}_4 = \vec{V}_4 - 1\vec{V}_1 - 2\vec{V}_2 - 0\vec{V}_3 = \begin{pmatrix}-2\\0\\3\\0\end{pmatrix}$$

$$\|\vec{X}_4\|^2 = \langle \vec{X}_4, \vec{X}_4 \rangle = (B\vec{X}_4)\cdot(B\vec{X}_4) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = 1$$

Then $\beta = \{\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{X}_4\}$ is an orthonormal basis.

3. *(20 pts)*

(a) The 2×2 matrix C below has eigenvectors $\overbrace{(2,3)}^{\vec{v}_1}$ and $\overbrace{(3,5)}^{\vec{v}_2}$. Find a fundamental set of solutions to the system $\vec{z}' = C\vec{z}$.

$$C = \begin{pmatrix} 11 & -6 \\ 15 & -8 \end{pmatrix}$$

$C\vec{v}_1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} = 2\vec{v}_1 \Rightarrow \lambda_1 = 2 \qquad C\vec{v}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 1\vec{v}_1 \Rightarrow \lambda_2 = 1$

By the theorem from class, a fundamental set of solutions is

$$\left\{ e^{2x}\begin{pmatrix} 2 \\ 3 \end{pmatrix}, e^{1x}\begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\}$$

(b) The matrices A and B are related by $A = QBQ^{-1}$, with

$$Q = \begin{pmatrix} 2/7 & 3/7 & -6/7 \\ -3/7 & 6/7 & 2/7 \\ 6/7 & 2/7 & 3/7 \end{pmatrix}$$

The system $\vec{y}' = A\vec{y}$ has a fundamental set of solutions consisting of

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \vec{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

Find a fundamental set of solutions to the system $\vec{w}' = B\vec{w}$ in terms of the functions $\vec{p}, \vec{q}, \vec{r}$.

$\vec{y}' = A\vec{y}$ $\qquad$ $\vec{y}$ solves $\vec{y}' = A\vec{y}$

$\vec{y}' = (QBQ^{-1})\vec{y}$ $\qquad$ iff

$(Q^{-1}\vec{y})' = B(Q^{-1}\vec{y})$ $\qquad$ $\vec{w} = Q^{-1}\vec{y}$ solves $\vec{w}' = B\vec{w}$.

So $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \{Q^{-1}\vec{p}, Q^{-1}\vec{q}, Q^{-1}\vec{r}\}$ is a

a fundamental set of solutions to $\vec{w}' = B\vec{w}$.

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Q is orthogonal, so

$$Q^{-1} = Q^T = \begin{pmatrix} 2/7 & -3/7 & 6/7 \\ 3/7 & 6/7 & 2/7 \\ -6/7 & 2/7 & 3/7 \end{pmatrix}$$

Then

$$\vec{w}_1 = Q^{-1}\vec{p} = \begin{pmatrix} 2/7\, p_1 - 3/7\, p_2 + 6/7\, p_3 \\ 3/7\, p_1 + 6/7\, p_2 + 2/7\, p_3 \\ -6/7\, p_1 + 2/7\, p_2 + 3/7\, p_3 \end{pmatrix}$$

$$\vec{w}_2 = Q^{-1}\vec{q} = \begin{pmatrix} 2/7\, q_1 - 3/7\, q_2 + 6/7\, q_3 \\ 3/7\, q_1 + 6/7\, q_2 + 2/7\, q_3 \\ -6/7\, q_1 + 2/7\, q_2 + 3/7\, q_3 \end{pmatrix}$$

$$\vec{w}_3 = Q^{-1}\vec{r} = \begin{pmatrix} 2/7\, r_1 - 3/7\, r_2 + 6/7\, r_3 \\ 3/7\, r_1 + 6/7\, r_2 + 2/7\, r_3 \\ -6/7\, r_1 + 2/7\, r_2 + 3/7\, r_3 \end{pmatrix}$$

4. *(24 pts)*

(a) Use the series definition of matrix exponentials to show that for every (constant) vector $\vec{v} \in \mathbb{R}^n$ and (constant) $n \times n$ matrix A, the vector-valued function $\vec{y}(x) = e^{xA}\vec{v}$ is a solution to the system $\vec{y}' = A\vec{y}$.

$$\left(e^{xA}\right)' = \left(1 + (xA) + \frac{(xA)^2}{2!} + \frac{(xA)^3}{3!} + \dots\right)'$$

$$= \left(A + \frac{2xA^2}{2!} + \frac{3x^2A^3}{3!} + \dots\right)$$

$$= A\left(1 + (xA) + \frac{(xA)^2}{2!} + \dots\right)$$

$$= A e^{xA}$$

Then

$$\vec{y}' = \left(e^{xA}\vec{v}\right)' = \left(e^{xA}\right)'\vec{v} = \left(Ae^{xA}\right)\vec{v} = A\vec{y}$$

(b) Show that if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$, then $\{e^{xA}\vec{v}_1, \dots, e^{xA}\vec{v}_n\}$ is a fundamental set of solutions to the system $\vec{y}' = A\vec{y}$.

The Wronskian is $W(x) = \det\begin{pmatrix} e^{xA}\vec{v}_1 & \cdots & e^{xA}\vec{v}_n \end{pmatrix}$

So $W(0) = \det\begin{pmatrix} e^{0}\vec{v}_1 & \cdots & e^{0}\vec{v}_n \end{pmatrix}$ and $e^0 = I$,

so $W(0) = \det\begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$. We know $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis so this det is $\neq 0$, and $W(0) \neq 0 \Rightarrow$ the list is linearly independant.

(c) Find a fundamental set of solutions to the system $\vec{y}' = A\vec{y}$, using the arithmetic below.

$$A = \underbrace{\begin{array}{c} \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \\ \begin{pmatrix} 2/49 & -3/49 & 6/49 \\ 3/49 & 6/49 & 2/49 \\ -6/49 & 2/49 & 3/49 \end{pmatrix} \end{array}}_{P} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}}_{J} \underbrace{\begin{pmatrix} 2 & 3 & -6 \\ -3 & 6 & 2 \\ 6 & 2 & 3 \end{pmatrix}}_{P^{-1}}$$

These matrices are inverses of each other so this is a conjugation, and this is in Jordan form, so it is the Jordan form for A. Rewriting as below shows that these columns are a Jordan basis $\mathcal{V}$.

$$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{V}}^{\mathcal{S}} [T]_{\mathcal{V}}^{\mathcal{V}} [I]_{\mathcal{S}}^{\mathcal{V}}$$

Then a fundamental set of solutions is

$$e^{xA}\vec{v}_1 = e^{2x}\vec{v}_1 = \frac{e^{2x}}{49}\begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$$

$$e^{xA}\vec{v}_2 = e^{3x}\vec{v}_2 = \frac{e^{3x}}{49}\begin{pmatrix} -3 \\ 6 \\ 2 \end{pmatrix}$$

$$e^{xA}\vec{v}_3 = e^{3x}\left(\vec{v}_3 + x\vec{v}_2\right) = \frac{e^{3x}}{49}\begin{pmatrix} 6-3x \\ 2+6x \\ 3+2x \end{pmatrix}$$

5. *(16 pts)*

(a) Use undetermined coefficients (do not use the integral formula) to find a particular solution to the system below for $k = 2$.

$$\vec{y}' = \overbrace{\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}}^{A} \vec{y} + \begin{pmatrix} 0 \\ e^{kx} \end{pmatrix}$$

We guess $\vec{y} = e^{kx}\vec{a}$. The equation becomes

$$ke^{kx}\vec{a} = Ae^{kx}\vec{a} + e^{kx}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(A - kI)\vec{a} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\vec{a} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow \vec{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

So our particular solution is $e^{2x}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(b) For what value(s) of k would the form of the guess for the particular solution in part (a) have to be changed? Show explicitly why the approach you used in part (a) would not work for that/those value(s) of k.

$A - kI = \begin{pmatrix} 3-k & 1 \\ 0 & 1-k \end{pmatrix}$ is singular for (eigenvalues) 1, 3.

For these values $A - kI$ is not invertible, and $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ also is not in the image so the system has no solutions!

So the form of our guess is wrong and would have to change.

(extra space for questions from other side)