EXAM 3

Math 216, 2020 Fall.

Name:____________________________________ NetID:____________ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

It is strongly advised that you use black pen only, since that will be most clear in scanning your work.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

(Scratch space. Nothing on this page will be graded!)

1. *(20 pts)*

   (a) Find the eigenvalues and eigenvectors of the matrix A below.

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

(extra space for questions from other side)

(b) List all of the Jordan forms that are possible given the results of part (a).

2. *(20 pts)* The inner product space V consists of the vector space $\mathbb{R}^4$, but using the inner product $\langle \vec{v}, \vec{w} \rangle = B\vec{v} \cdot B\vec{w}$ with

$$B = \begin{pmatrix} 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

(a) Confirm that the first three vectors listed in the basis α (below) for V below are orthonormal.

$$\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \\ -2 \end{pmatrix} \right\}$$

(b) Perform Gram-Schmidt orthonormalization on the basis α to find an orthonormal basis for V.

3. *(20 pts)*

(a) The 2×2 matrix C below has eigenvectors $(2, 3)$ and $(3, 5)$. Find a fundamental set of solutions to the system $\vec{z}' = C\vec{z}$.

$$C = \begin{pmatrix} 11 & -6 \\ 15 & -8 \end{pmatrix}$$

(b) The matrices A and B are related by $A = QBQ^{-1}$, with

$$Q = \begin{pmatrix} 2/7 & 3/7 & -6/7 \\ -3/7 & 6/7 & 2/7 \\ 6/7 & 2/7 & 3/7 \end{pmatrix}$$

The system $\vec{y}' = A\vec{y}$ has a fundamental set of solutions consisting of

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \vec{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

Find a fundamental set of solutions to the system $\vec{w}' = B\vec{w}$ in terms of the functions $\vec{p}, \vec{q}, \vec{r}$.

(extra space for questions from other side)

4. *(24 pts)*

(a) Use the series definition of matrix exponentials to show that for every (constant) vector $\vec{v} \in \mathbb{R}^n$ and (constant) $n \times n$ matrix A, the vector-valued function $\vec{y}(x) = e^{xA}\vec{v}$ is a solution to the system $\vec{y}' = A\vec{y}$.

(b) Show that if $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$, then $\{e^{xA}\vec{v}_1, \ldots, e^{xA}\vec{v}_n\}$ is a fundamental set of solutions to the system $\vec{y}' = A\vec{y}$.

(c) Find a fundamental set of solutions to the system $\vec{y}' = A\vec{y}$, using the arithmetic below.

$$A = \begin{pmatrix} 2/49 & -3/49 & 6/49 \\ 3/49 & 6/49 & 2/49 \\ -6/49 & 2/49 & 3/49 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 & -6 \\ -3 & 6 & 2 \\ 6 & 2 & 3 \end{pmatrix}$$

5. *(16 pts)*

(a) Use undetermined coefficients (do not use the integral formula) to find a particular solution to the system below for $k = 2$.

$$\vec{y}' = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \vec{y} + \begin{pmatrix} 0 \\ e^{kx} \end{pmatrix}$$

(b) For what value(s) of k would the form of the guess for the particular solution in part (a) have to be changed? Show explicitly why the approach you used in part (a) would not work for that/those value(s) of k.

(extra space for questions from other side)