EXAM 2

Math 216, 2020 Fall.

Name: Solutions NetID:__________ Student ID:__________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

It is strongly advised that you use black pen only, since that will be most clear in scanning your work.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

(Scratch space. Nothing on this page will be graded!)

1. *(20 pts)* Find a fundamental set of real solutions to the differential equation $L(y) = 0$ with characteristic polynomial

$$p(\lambda) = \underbrace{(\lambda^2 - 5\lambda + 6)}_{(\lambda-2)(\lambda-3)=}(\lambda - 3)^2\underbrace{(\lambda^3 + 8)}_{\text{roots are: } -2,\, -2e^{2\pi i/3},\, -2e^{4\pi i/3}}\Big(\lambda - (2 - i)\Big)^2\Big(\lambda - (2 + i)\Big)^2$$

$= -2,\ 1-i\sqrt{3},\ 1+i\sqrt{3}$

So $p(\lambda) = (\lambda-2)(\lambda-3)^3(\lambda+2)\left(\lambda-(1-i\sqrt{3})\right)\left(\lambda-(1+i\sqrt{3})\right)\left(\lambda-(2-i)\right)^2\left(\lambda-(2+i)\right)^2$

By theorem from the lectures, a fundamental set of real solutions is:

$$\left\{e^{2x}, e^{3x}, xe^{3x}, x^2e^{3x}, e^{-2x}, e^x\cos(x\sqrt{3}), e^x\sin(x\sqrt{3}), e^{2x}\cos x, e^{2x}\sin x, xe^{2x}\cos x, xe^{2x}\sin x\right\}$$

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2. *(20 pts)* The following is a fundamental set of solutions to the differential equation $L(y) = 0$:

$$\{e^{3x}, xe^{3x}, e^{4x}, e^{2x}\cos 3x, e^{2x}\sin 3x, xe^{2x}\cos 3x, xe^{2x}\sin 3x, \}$$

Find the form (you do not have to evaluate the coefficients!) of a particular solution to the differential equation

$$L(y) = \underbrace{5x^2e^{2x}}_{g_1} \underbrace{- 6e^{3x}}_{g_2} + \underbrace{xe^{2x}\cos 3x}_{g_3}$$

$$\Rightarrow p(\lambda) = (\lambda-3)^2(\lambda-4)(\lambda-(2+3i))^2(\lambda-(2-3i))^2$$

g_1: $r = a+bi = 2$, not a root of p.

So $y_{p_1} = (c_2x^2 + c_1x + c_0)e^{2x}$

g_2: $r = a+bi = 3$, is a root of p, with $m=2$.

So $y_{p_2} = dx^2(e^{3x})$

g_3: $r = a+bi = 2+3i$, is a root of p, with $m=2$.

So $y_{p_3} = x^2(e_1x+e_0)e^{2x}\cos 3x + x^2(f_1x+f_0)e^{2x}\sin 3x$

Then

$$\begin{aligned} y_p &= y_{p_1} + y_{p_2} + y_{p_3} \\ &= (c_2x^2+c_1x+c_0)e^{2x} + dx^2(e^{3x}) \\ &\quad + x^2(e_1x+e_0)e^{2x}\cos 3x + x^2(f_1x+f_0)e^{2x}\sin 3x \end{aligned}$$

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3. *(20 pts)* The differential equation $L(z) = 3e^{2it}$ has a particular solution $z_p = (2+2i)e^{2it}$. Find a particular solution to the equation $L(y) = \cos(2t)$, and identify the gain and the phase shift in this equation. (Show all of the algebra needed to justify your results!)

$$z_p = (2+2i)e^{2it} = \left(2\sqrt{2}\, e^{\pi i/4}\right)e^{2it} = 2\sqrt{2}\, e^{i(2t+\pi/4)}$$

$$3e^{2it} = 3\cos 2t + i\,3\sin 2t$$

$$\Rightarrow \cos 2t = \frac{1}{3}\mathrm{Re}\left(3e^{2it}\right)$$

$$\Rightarrow \begin{pmatrix}\text{sol. to}\\ L(y)=\cos 2t\end{pmatrix} = \frac{1}{3}\mathrm{Re}\begin{pmatrix}\text{sol. to}\\ L(z)=3e^{2it}\end{pmatrix}$$

$$\Rightarrow y = \frac{1}{3}\mathrm{Re}\left(2\sqrt{2}\, e^{i(2t+\pi/4)}\right)$$

$$= \frac{2\sqrt{2}}{3}\cos\left(2t+\pi/4\right)$$

$$\Rightarrow \text{gain} = \frac{2\sqrt{2}}{3}, \quad \text{phase shift} = \frac{-\pi}{4}$$

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4. *(20 pts)* The list $\alpha = \{v_1, v_2, v_3\}$ is a basis for the vector space V. The "coordinate function" $f : V \to \mathbb{R}^3$ is defined by $f(v) = [v]_\alpha$. Show (directly from the definition) that f is a linear transformation. (This result was claimed in class, and used in question 5 from the first exam this term; you may NOT cite those references here.)

We must show that $f(ax+by) = af(x) + bf(y)$.

First we compute directly:

$$\left.\begin{aligned} x &= c_1v_1 + c_2v_2 + c_3v_3 \\ y &= d_1v_1 + d_2v_2 + d_3v_3 \end{aligned}\right\} \Rightarrow ax+by = a(c_1v_1+c_2v_2+c_3v_3) + b(d_1v_1+d_2v_2+d_3v_3)$$

$$= (ac_1+bd_1)v_1 + (ac_2+bd_2)v_2 + (ac_3+bd_3)v_3$$

This tells us that

$$[ax+by]_\alpha = \begin{pmatrix} ac_1+bd_1 \\ ac_2+bd_2 \\ ac_3+bd_3 \end{pmatrix} = a\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + b\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = a[x]_\alpha + b[y]_\alpha$$

$$f(ax+by) = af(x) + bf(y)$$

as required.

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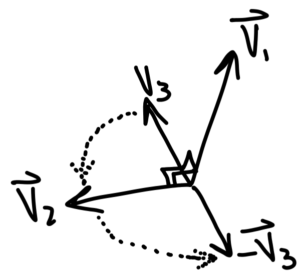
5. *(20 pts)* The line L goes through the origin and is parallel to $\vec{v}_1 = (1, 2, 2)$. Find the matrix M that rotates points in $\mathbb{R}^3$ a quarter turn counterclockwise around L (as seen from above). *(Hint #1: $\vec{v}_2 = (2, 1, -2)$ and $\vec{v}_3 = (2, -2, 1)$ might help form a convenient basis. Hint #2: If the columns of a matrix are all perpendicular, then the product with its transpose is diagonal.)*

Dot products show $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are all perpendicular with magnitude 3.

$\vec{V}_2 \times \vec{V}_3 = (-3, -6, -6) = -3\vec{v}_1$

points down, so the rotation as described gives



$$R(\vec{v}_1) = \vec{v}_1 \qquad R(\vec{v}_2) = -\vec{v}_3 \qquad R(\vec{v}_3) = \vec{v}_2$$

Using the basis $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, we have

$$[R]_\alpha^\alpha = \begin{pmatrix} | & | & | \\ [R(\vec{v}_1)]_\alpha & [R(\vec{v}_2)]_\alpha & [R(\vec{v}_3)]_\alpha \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

To compute $M = [R]_\delta^\delta$, we need $[I]_\alpha^\delta$ and $[I]_\delta^\alpha$.

From the givens we have

$$[I]_\alpha^\delta = \begin{pmatrix} | & | & | \\ [\vec{v}_1]_\delta & [\vec{v}_2]_\delta & [\vec{v}_3]_\delta \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

To compute $[I]_\delta^\alpha = \left([I]_\alpha^\delta\right)^{-1}$, we compute

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as suggested by Hint #2 :

$$\underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}}_{[I]^\beta_\alpha}\underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}}_{\left([I]^\beta_\alpha\right)^T} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}}_{[I]^\beta_\alpha}\underbrace{\frac{\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}}{9}}_{\left([I]^\beta_\alpha\right)^{-1} = [I]^\alpha_\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $M = [R]^\beta_\beta = [I]^\beta_\alpha [R]^\alpha_\alpha [I]^\alpha_\beta$

$$= \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\Big/9$$

$$= \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix}\Big/9$$

$$= \begin{pmatrix} 1 & -4 & 8 \\ 8 & 4 & 1 \\ -4 & 7 & 4 \end{pmatrix}\Big/9$$