EXAM 1

Math 216, 2020 Fall.

Name: Solutions NetID: Student ID:

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

It is strongly advised that you use black pen only, since that will be most clear in scanning your work.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature:

(Scratch space. Nothing on this page will be graded!)

1. *(20 pts)*

(a) Bob is doing a row operation of a matrix. At some point he has a matrix with rows R_1, R_2, R_3, and he is contemplating as his next "step" to replace the first row with $R_1 + 2R_2$, the second row with $R_2 + 2R_3$, and the third row with $4R_3 - R_1$. What can you tell Bob about the advisability of this?

$$\underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 4 \end{pmatrix}}_{\text{equivalent matrix } F} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} R_1+2R_2 \\ R_2+2R_3 \\ 4R_3-R_1 \end{pmatrix} \underbrace{\begin{matrix} 1(1)+2(2) \\ 1(2)+2(3) \\ -1(1)+4(3) \end{matrix}}_{\text{proposed "step"}}$$

$\det F = 1(1\cdot4-2\cdot0) + (-1)(2\cdot2-0\cdot1) = 0$, so F is not nonsingular, thus not a product of elementary matrices, so Bob's "step" is not a combination of row operations!

(b) Suppose that the matrix A has the existence property but does not have the uniqueness property. Explain in detail how you know that A^T cannot have the existence property. *(Hint: What can you conclude about the number of rows and the number of columns?)*

Say A is $m \times n$. The givens tell us

$$\underbrace{m = \text{rank}(A)}_{\text{existence}} \underbrace{< n}_{\text{no uniqueness}}$$

This means that A^T, being $n \times m$, has more rows than columns, and so also that it has more rows than pivots.

So $\text{rref}(A^T)$ must have a row with no pivot, and thus A^T cannot have the existence property.

(extra space for questions from other side)

2. *(20 pts)*

(a) The 3×3 matrix $B = A^{-1}$ has rows B_1, B_2, B_3, and columns $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$. Find (by describing either its rows or columns) the inverse of $C = PA$, where

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$C^{-1} = (PA)^{-1} = A^{-1}P^{-1} = BP^{-1}$$

$$= \begin{pmatrix} | & | & | \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} | & | & | \\ \vec{b}_3 & \vec{b}_1 & \vec{b}_2 \\ | & | & | \end{pmatrix}$$

(b) Find elementary matrices $E_1, \ldots, E_k$ such that

$$E_1 \cdots E_k = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} = A$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$$

$$\begin{array}{ll} \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix} & \begin{array}{l} 1 \\ 2 - 2\,1 \end{array} \end{array} \Bigg\} F_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\begin{array}{ll} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} & \begin{array}{l} 1 \\ -2 \end{array} \end{array} \Bigg\} F_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{array}{ll} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{array}{l} 1 - 3\,2 \\ 2 \end{array} \end{array} \Bigg\} F_3 = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

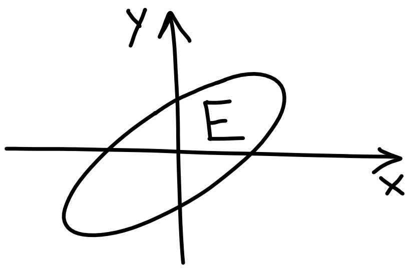
$$F_3 F_2 F_1 A = I$$

$$A = F_1^{-1} F_2^{-1} F_3^{-1}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{E_1} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{E_2} \underbrace{\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}}_{E_3}$$

(extra space for questions from other side)

3. *(16 pts)* The curve C in the xy-plane has equation $(2x-3y)^2+(x+y)^2=1$. Find (using ideas from this course!) the area enclosed by C. *(Hint: Find a matrix that relates this to the unit circle in the uv-plane.)*

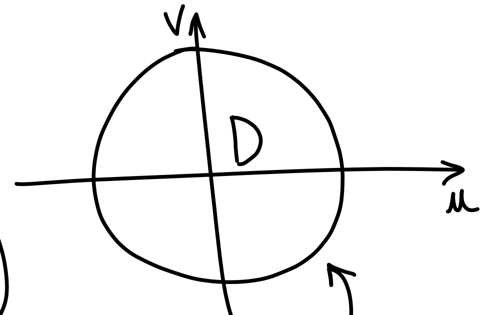


$(2x-3y)^2+(x+y)^2=1$

$2x-3y=u$

$x+y=v$

$$\underbrace{\begin{pmatrix}2 & -3\\ 1 & 1\end{pmatrix}}_{A}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}u\\ v\end{pmatrix}$$



$u^2+v^2=1$

A pair (x,y) satisfying the xy equation, times A gives a pair (u,v) satisfying the uv equation (same arithmetic!).

So D is the image of E. Interpreting in terms of areas and stretching factors,

$$(\text{area of } E)(\text{stretching factor of } A) = (\text{area of } D)$$

$$(\text{area of } E)(|\det A|) = \pi$$

$$(\text{area of } E)(5) = \pi$$

$$(\text{area of } E) = \pi/5$$

(extra space for questions from other side)

4. *(24 pts)* We will say that a vector space X "splits" the spaces V and W if either $V \subsetneq X \subsetneq W$ or $W \subsetneq X \subsetneq V$. ("$P \subsetneq Q$" means that P is a subspace of Q and is not equal to Q.)

(a) Is there a vector space C that splits $A = \mathbb{R}^3$ and $B = \{\text{the } x\text{-axis in } \mathbb{R}^3\}$? If so, find it (you don't have to prove your claim); if not, explain why it can't exist.

C can be the xy-plane in $\mathbb{R}^3$ (or any plane in $\mathbb{R}^3$ containing the x-axis.)

(b) Is there a vector space F that splits $D = \text{span}\{\sin x - \cos x,\ 3\sin x + 2\cos x,\ 2\sin x - \cos x\}$ and $E = \text{span}\{\sin x,\ \cos x,\ e^x,\ e^{-x}\}$? If so, find it (you don't have to prove your claim); if not, explain why it can't exist.

We first observe that $D = \text{span}\{\sin x, \cos x\}$.

Then we choose $F = \text{span}\{\sin x, \cos x, e^x\}$.

(c) Suppose that $V \subsetneq W$ are finite-dimensional. On what condition does there exist a vector space that splits them? (You don't have to prove your claim.)

We need $\dim W \geq \dim V + 2$.

(extra space for questions from other side)

5. *(20 pts)* In this question we consider vectors in the vector space

$$V = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \middle| f_1, f_2 \text{ are real-valued functions on } \mathbb{R} \right\}$$

(a) Show that $\alpha = \left\{ \begin{pmatrix} \sin x \\ 0 \end{pmatrix}, \begin{pmatrix} \cos x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sin x \end{pmatrix}, \begin{pmatrix} 0 \\ \cos x \end{pmatrix} \right\}$ is a basis for its span W.

A relation on α would be

$$c_1 \begin{pmatrix} \sin x \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \cos x \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ \sin x \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ \cos x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 \sin x + c_2 \cos x \\ c_3 \sin x + c_4 \cos x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} c_1 \sin x + c_2 \cos x = 0 & \leftarrow 1 \\ c_3 \sin x + c_4 \cos x = 0 & \leftarrow 2 \end{cases}$$

We know $\{\sin x, \cos x\}$ is l.i.; so $1 \Rightarrow c_1, c_2 = 0$, $2 \Rightarrow c_3, c_4 = 0$

So the only relation is the trivial relation and α is l.i..

(b) Decide if $\beta = \left\{ \begin{pmatrix} \sin x - \cos x \\ \sin x + \cos x \end{pmatrix}, \begin{pmatrix} \cos x \\ \sin x + \cos x \end{pmatrix}, \begin{pmatrix} \sin x + \cos x \\ \sin x \end{pmatrix} \right\}$ is linearly independent.

(Show how you use that $[av + bw]_{\mathcal{V}} = a[v]_{\mathcal{V}} + b[w]_{\mathcal{V}}$, and $([v]_{\mathcal{V}} = \vec{0}) \Leftrightarrow (v = 0)$.)

A relation on β is

$$c_1 \begin{pmatrix} \sin x - \cos x \\ \sin x + \cos x \end{pmatrix} + c_2 \begin{pmatrix} \cos x \\ \sin x + \cos x \end{pmatrix} + c_3 \begin{pmatrix} \sin x + \cos x \\ \cos x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[c_1 \begin{pmatrix} \sin x - \cos x \\ \sin x + \cos x \end{pmatrix} + c_2 \begin{pmatrix} \cos x \\ \sin x + \cos x \end{pmatrix} + c_3 \begin{pmatrix} \sin x + \cos x \\ \cos x \end{pmatrix} \right]_\alpha = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right]_\alpha$$

$$c_1 \left[\begin{pmatrix} \sin x - \cos x \\ \sin x + \cos x \end{pmatrix} \right]_\alpha + c_2 \left[\begin{pmatrix} \cos x \\ \sin x + \cos x \end{pmatrix} \right]_\alpha + c_3 \left[\begin{pmatrix} \sin x + \cos x \\ \cos x \end{pmatrix} \right]_\alpha = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right]_\alpha$$

(extra space for questions from other side)

$$c_1\begin{pmatrix}1\\-1\\1\\1\end{pmatrix}+c_2\begin{pmatrix}0\\1\\1\\1\end{pmatrix}+c_3\begin{pmatrix}1\\1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1&0&1&0\\-1&1&1&0\\1&1&0&0\\1&1&1&0\end{array}\right)$$

$$\left(\begin{array}{ccc|c}1&0&1&0\\0&1&2&0\\0&1&-1&0\\0&1&0&0\end{array}\right)\begin{array}{l}1\\2+1\\3-1\\4-1\end{array}$$

$$\left(\begin{array}{ccc|c}1&0&1&0\\0&1&2&0\\0&0&1&0\\0&0&-2&0\end{array}\right)\begin{array}{l}1\\2\\(3-2)/-3\\4-2\end{array}$$

$$\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right)\begin{array}{l}1-3\\2-2\,3\\3\\4+2\,3\end{array}$$

There is a pivot in every column of the rref.

So $c_1=0$, $c_2=0$, $c_3=0$ is the only solution.

So β has no significant relations, and thus β is linearly independent.