EXAM 3

Math 216, 2020 Spring, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(18 pts)* The basis $\mathcal{V}$ consists of $\vec{v}_1 = (6, 2, 3)$, $\vec{v}_2 = (-3, 6, 2)$, and $\vec{v}_3 = (2, 3, -6)$. The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the reflection through the plane P spanned by the vectors $\vec{v}_1$ and $\vec{v}_2$.

(a) Find the change of basis matrix $[I]^{\mathcal{S}}_{\mathcal{V}}$.

$$[I]^{\mathcal{S}}_{\mathcal{V}} = \begin{pmatrix} [\vec{v}_1]_{\mathcal{S}} & [\vec{v}_2]_{\mathcal{S}} & [\vec{v}_3]_{\mathcal{S}} \end{pmatrix} = \begin{pmatrix} 6 & -3 & 2 \\ 2 & 6 & 3 \\ 3 & 2 & -6 \end{pmatrix} = P$$

(b) Find the change of basis matrix $[I]^{\mathcal{V}}_{\mathcal{S}}$. *(Hint: What feature of $\mathcal{V}$ will help with this?)*

$\mathcal{V}$ is orthogonal, and all vectors have magnitude 7, so $P/7$ is an orthogonal matrix. Then

$$7P^{-1} = \left(\frac{P}{7}\right)^{-1} = \left(\frac{P}{7}\right)^{T} = \frac{P^T}{7}$$

So

$$P^{-1} = \frac{P^T}{49} = \begin{pmatrix} 6 & 2 & 3 \\ -3 & 6 & 2 \\ 2 & 3 & -6 \end{pmatrix} \Big/ 49$$

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(c) Use the results from the previous two parts to find the matrix $[T]_{\mathcal{S}}^{\mathcal{S}}$.

$$\left.\begin{aligned} T(\vec{v}_1) &= \vec{v}_1 \\ T(\vec{v}_2) &= \vec{v}_2 \end{aligned}\right\} \text{because these vectors are in the plane of reflection}$$

$$T(\vec{v}_3) = -\vec{v}_3 \quad \Big\} \text{because } \vec{v}_3 \text{ is } \perp \text{ to the plane of reflection}$$

$$\text{So } [T]_{\mathcal{V}}^{\mathcal{V}} = \left([T(\vec{v}_1)]_{\mathcal{V}} \; [T(\vec{v}_2)]_{\mathcal{V}} \; [T(\vec{v}_3)]_{\mathcal{V}} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Then

$$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{V}}^{\mathcal{S}} [T]_{\mathcal{V}}^{\mathcal{V}} [I]_{\mathcal{S}}^{\mathcal{V}}$$

$$= P \, [T]_{\mathcal{V}}^{\mathcal{V}} \, P^{-1}$$

$$= \begin{pmatrix} 6 & -3 & 2 \\ 2 & 6 & 3 \\ 3 & 2 & -6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 6 & 2 & 3 \\ -3 & 6 & 2 \\ 2 & 3 & -6 \end{pmatrix} / 49$$

$$= \begin{pmatrix} 6 & -3 & 2 \\ 2 & 6 & 3 \\ 3 & 2 & -6 \end{pmatrix} \begin{pmatrix} 6 & 2 & 3 \\ -3 & 6 & 2 \\ -2 & -3 & 6 \end{pmatrix} / 49$$

$$= \begin{pmatrix} 41 & -12 & 24 \\ -12 & 31 & 36 \\ 24 & 36 & -23 \end{pmatrix} / 49$$

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2. *(16 pts)* The non-diagonalizable matrix A has eigenvalues 2 and 3 with multiplicities 1 and 2, respectively. It can be put into Jordan form by way of the basis $\mathcal{V}$ consisting of $\vec{v}_1 = (1,0,1)$, $\vec{v}_2 = (0,1,2)$, and $\vec{v}_3 = (0,0,2)$ (in this order).

(a) Given only the information above, what are the possible Jordan forms for A?

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(b) Suppose we know that $\vec{v}_2$ is an eigenvector. Find the only possible Jordan form for A.

$\vec{v}_2$ is an eigenvector iff the 2nd column of the Jordan form is diagonal. So the only possibility is

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

(c) Find the matrix A itself.

$$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{V}}^{\mathcal{S}} [T]_{\mathcal{V}}^{\mathcal{V}} [I]_{\mathcal{S}}^{\mathcal{V}}$$

$$A = P \quad J \quad P^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 2 & 6 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ -2 & -2 & 4 \end{pmatrix}$$

(over)

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Scratch for finding $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}^{-1}$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -1 & \frac{1}{2} \end{array}\right) \begin{array}{l} (1) \\ (2) \\ ((3) - (1) - 2(2))/2 \end{array}$$

3. *(16 pts)* The vectors $v_1, \ldots, v_n$ in the inner product space V are known to be orthonormal. Show that these vectors are linearly independent. *(Hint: Write a relation, and take clever inner products.)*

Suppose we have a relation

$$c_1\vec{v}_1 + \ldots + c_i\vec{v}_i + \ldots + c_n\vec{v}_n = \vec{0}$$

Taking inner product of both sides with $\vec{v}_i$ gives

$$\langle c_1\vec{v}_1 + \ldots + c_i\vec{v}_i + \ldots + c_n\vec{v}_n , \vec{v}_i \rangle = 0$$

The inner product is linear in the left entry, so we have

$$c_1\langle \vec{v}_1, \vec{v}_i\rangle + \ldots + c_i\langle \vec{v}_i, \vec{v}_i\rangle + \ldots + c_n\langle \vec{v}_n, \vec{v}_i\rangle = 0$$

By orthonormality these inner products are all zero except for $\langle \vec{v}_i, \vec{v}_i\rangle = 1$. So we have

$$1 \cdot c_i = 0$$

and thus $c_i = 0$. This is true for all i, so the relation must be trivial and thus the list is linearly independent.

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4. *(18 pts)* Find the complete set of solutions to

$$\vec{y}' = \begin{pmatrix} 17 & -12 \\ 20 & -14 \end{pmatrix} \vec{y} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Homogeneous solutions:

$$p(\lambda) = \det \begin{pmatrix} 17-\lambda & -12 \\ 20 & -14-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$$

So we have eigenvalues 1, 2.

$\lambda = 1 \Rightarrow A - \lambda I = \begin{pmatrix} 16 & -12 \\ 20 & -15 \end{pmatrix}$ has rank=1 $\Rightarrow$ dim NS=1

so the only eigenvector is $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

$\lambda = 2 \Rightarrow A - \lambda I = \begin{pmatrix} 15 & -12 \\ 20 & -16 \end{pmatrix}$ has rank=1 $\Rightarrow$ dim NS=1

so the only eigenvector is $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

So the homogeneous solutions are $\left\{ e^{1x}\begin{pmatrix} 3 \\ 4 \end{pmatrix}, e^{2x}\begin{pmatrix} 4 \\ 5 \end{pmatrix} \right\}$

Particular solution:

We guess $\vec{y}_p = \vec{a}$, the equation becomes $\vec{0} = A\vec{a} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

So $\vec{a} = A^{-1}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}\Big/ 2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$.

General solution

$$\vec{y} = \vec{y}_H + \vec{y}_p = c_1 e^{1x}\begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 e^{2x}\begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

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5. *(16 pts)* Use the basis consisting of the vectors $\vec{v}_1 = (1,1)$ and $\vec{v}_2 = (0,1)$ to derive a back-solvable system $\vec{z}' = B\vec{z}$ (do not solve!) whose solutions would allow for solving the system

$$\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= -y_1 + 3y_2 \end{aligned}$$

and derive a formula for solutions $\vec{y}(t)$ in terms of those solutions $\vec{z}(t)$.

$\vec{y}' = A\vec{y}$, $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} = A = PBP^{-1}$

$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{B}}^{\mathcal{S}} [T]_{\mathcal{B}}^{\mathcal{B}} [I]_{\mathcal{S}}^{\mathcal{B}}$

So we choose

$$\begin{aligned} B &= P^{-1}AP \\ &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{aligned}$$

Then

$$\begin{aligned} \vec{y}' &= A\vec{y} \\ &= PBP^{-1}\vec{y} \\ P^{-1}\vec{y}' &= BP^{-1}\vec{y} \\ \vec{z}' &= B\vec{z} \end{aligned}$$

rewrite $\vec{z} = P^{-1}\vec{y}$

so $\vec{y} = P\vec{z}$

And $\vec{z}' = B\vec{z}$ is

$$\begin{aligned} z_1' &= 2z_1 + 1z_2 \\ z_2' &= 2z_2 \end{aligned}$$

which is back-solvable.

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6. *(16 pts)*

(a) If the equation below were converted to a first order system $\vec{y}' = A\vec{y}$, what would be the characteristic polynomial of the coefficient matrix A?

$$y''' - 3y'' + 2y = 0$$

It would be the same as for the original equation,

$$p(\lambda) = \lambda^3 - 3\lambda^2 + 2$$

(b) Find a first order system of linear differential equations whose solutions would allow also for identifying the solutions to the system below.

$$\begin{aligned} y_1''' &= y_1 - 2y_2 + 3y_1' + 4y_2'' \\ y_2''' &= 5y_1 + 6y_1' + 7y_1'' - 8y_2 \end{aligned}$$

Set
$$\begin{aligned} y_1 &= u_1 \\ y_1' &= u_2 \\ y_1'' &= u_3 \\ y_2 &= u_4 \\ y_2' &= u_5 \\ y_2'' &= u_6 \end{aligned}$$

Then

$$\begin{aligned} u_1' &= 0u_1 + 1u_2 + 0u_3 + 0u_4 + 0u_5 + 0u_6 \\ u_2' &= 0u_1 + 0u_2 + 1u_3 + 0u_4 + 0u_5 + 0u_6 \\ u_3' &= 1u_1 + 3u_2 + 0u_3 - 2u_4 + 0u_5 + 4u_6 \\ u_4' &= 0u_1 + 0u_2 + 0u_3 + 0u_4 + 1u_5 + 0u_6 \\ u_5' &= 0u_1 + 0u_2 + 0u_3 + 0u_4 + 0u_5 + 1u_6 \\ u_6' &= 5u_1 + 6u_2 + 7u_3 - 8u_4 + 0u_5 + 0u_6 \end{aligned}$$

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