EXAM 2

Math 216, 2020 Spring, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything near the staple – this will be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(16 pts)*

(a) What is the value $w(1)$ of the Wronskian of the list of functions below defined on $[-1, 1]$? *(Reminder – be sure to explain all of your reasoning!)*

$$\sin(x), \csc\left(x - \frac{\pi}{2}\right), \sin\left(x - \frac{\pi}{3}\right), \cos\left(x - \frac{\pi}{4}\right)$$

The 1st, 3rd, and 4th of these are all in span $\{\sin x, \cos x\}$, which is 2-dimensional;

So the list is linearly dependent, and thus

$w(1) = 0$.

(b) It is known that the Wronskian of the linearly independent list y_1, y_2, y_3, y_4 is zero for all values of x. Does this information allow us to draw any additional conclusions, and if so, what is it that we can conclude?

If these functions were all analytic, then the given $w(x)=0$ would imply the list was linearly dependent.

Since we know this is not true, we must conclude that at least one of these functions is not analytic.

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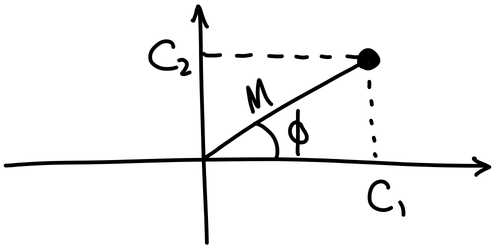
2. *(18 pts)* Let V be the collection of all functions of the form $f(x) = M\cos(x - \phi)$, where M and ϕ can be any real values.

(a) Show that V is a vector space. *(Hint: Rewrite it as a span and cite a theorem about spans.)*

$$M\cos(x-\phi) = (M\cos\phi)\cos x + (M\sin\phi)\sin x$$

With $C_1 = M\cos\phi$, $C_2 = M\sin\phi$, we see that V is



$\text{span}\{\cos x, \sin x\}$, and by a theorem from class we know this is a subspace, and therefore a vector space.

(b) Identify a basis for V and compute the coordinates with respect to that basis of $g(x) = \cos(x - \pi/3)$.

$\alpha = \{\cos x, \sin x\}$ is a natural basis for V by the algebra above.

$$g = \cos\left(x - \frac{\pi}{3}\right) = \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x$$

So $[g]_\alpha = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$.

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3. *(16 pts)* Find a fundamental set of real solutions to the differential equation $L(y) = 0$ whose characteristic polynomial is $p(\lambda) = (\lambda - 4)^3(\lambda^2 + 4)^2(\lambda^3 + 6\lambda^2 + 11\lambda + 6)$.

(λ^2+4) factors as $(\lambda+2i)(\lambda-2i)$.

$\lambda^3+6\lambda^2+11\lambda+6$ has possible rational roots $\pm1, \pm2, \pm3, \pm6$.

$p(-1)=0 \Rightarrow \lambda+1$ is a factor.

$$\begin{array}{r|l} & \lambda^2+5\lambda+6 \\ \lambda+1 & \lambda^3+6\lambda^2+11\lambda+6 \\ & \underline{\lambda^3+1\lambda^2} \\ & \quad 5\lambda^2+11\lambda+6 \\ & \quad \underline{5\lambda^2+5\lambda} \\ & \qquad 6\lambda+6 \\ & \qquad \underline{6\lambda+6} \\ & \qquad\quad 0 \end{array}$$

And

$\lambda^2+5\lambda+6 = (\lambda+2)(\lambda+3)$

So $p(\lambda) = (\lambda-4)^3(\lambda+2i)^2(\lambda-2i)^2(\lambda+1)(\lambda+2)(\lambda+3)$

So a f.s.s. is

$$\{e^{4x}, xe^{4x}, x^2e^{4x}, \cos 2x, \sin 2x, x\cos 2x, x\sin 2x, e^{-1x}, e^{-2x}, e^{-3x}\}$$

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4. *(16 pts)* Find the form of a particular solution to the differential equation

$$y'' - 4y' + 29y = \underset{\overset{\uparrow}{r=0}}{3x} + \underbrace{xe^{2x}\cos(5x)}_{\overset{\uparrow}{r=2+5i}} - \underbrace{x^2e^{4x}\sin(5x)}_{\overset{\uparrow}{r=4+5i}}$$

$r=2+5i$ ← This is a root of p, with $m=1$.

$$p(\lambda) = \lambda^2 - 4\lambda + 29$$

$$= (\lambda-2)^2 + 5^2$$

has roots $2 \pm 5i$

Particular solution has the form

$$y_p = (Ax+B) + x\left((c_1x+c_0)e^{2x}\cos 5x + (d_1x+d_0)e^{2x}\sin 5x\right)$$

$$+ (e_2x^2+e_1x+e_0)e^{4x}\cos 5x + (f_2x^2+f_1x+f_0)e^{4x}\sin 5x$$

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5. *(16 pts)* A mass on a spring in a resistive medium is currently critically damped. Using certain units, it is known that the mass is 3 and the spring constant is 27.

(a) What is the friction coefficient of this mass in this resistive medium?

$p(\lambda) = m\lambda^2 + f\lambda + k$ is critically damped

with $f^2 - 4mk = 0$.

So $f = \sqrt{4mk} = \sqrt{4 \cdot 3 \cdot 27} = 18$

(b) What is the (smallest) increase in the mass that would result in a (decaying) oscillation with frequency equal to $1/2\pi$?

$p(\lambda) = m\lambda^2 + 18\lambda + 27$ has roots

$$r = \frac{-18 \pm \sqrt{324 - 108m}}{2m} = \frac{-9}{m} \pm i \underbrace{\frac{\sqrt{27m - 81}}{m}}_{\omega}$$

$$\text{freq} = \frac{1}{2\pi} \Rightarrow \omega = 1 = \frac{\sqrt{27m - 81}}{m}$$

$$1m^2 - 27m + 81 = 0$$

$$m = \frac{27 \pm \sqrt{729 - 324}}{2}$$

$$= \frac{27 \pm \sqrt{405}}{2}$$

So the smallest increase in mass would be

$$\frac{27 - \sqrt{405}}{2} - 3.$$

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6. *(18 pts)* Let V be the vector space of solutions to the differential equation $y'' - y' + y = 0$.

(a) Show that $T : V \to \mathbb{R}^2$ defined by $T(f) = \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}$ is a linear transformation.

The domain and target are vector spaces, and

$$T(c_1f_1 + c_2f_2) = \begin{pmatrix} (c_1f_1+c_2f_2)(0) \\ (c_1f_1+c_2f_2)(0) \end{pmatrix} = \begin{pmatrix} c_1 f_1(0) + c_2 f_2(0) \\ c_1 f_1'(0) + c_2 f_2'(0) \end{pmatrix}$$

$$= c_1 \begin{pmatrix} f_1(0) \\ f_1'(0) \end{pmatrix} + c_2 \begin{pmatrix} f_2(0) \\ f_2'(0) \end{pmatrix} = c_1T(f_1) + c_2T(f_2)$$

So T is a linear transformation.

(b) What theorem allows us to conclude that T is bijective? Check that the conditions of this theorem apply to the given differential equation.

The existence + uniqueness theorem. Conditions:

1) the coefficient functions $(1, -1, 1)$ are continuous;
2) the right side (0) is continuous;
3) the lead coefficient (1) is never zero.

(c) What does the bijectivity of T allow us to conclude about the dimension of V? (Explain!)

T being a bijective linear transformation allows us to conclude that its domain and target have the same dimensions. The target $(\mathbb{R}^2)$ is clearly 2-dimensional, so the domain (set of solutions) must be 2-dimensional.

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