EXAM 1

Math 216, 2020 Spring, Clark Bray.

Name: Solutions ____________ Section: ______ Student ID: ____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything near the staple – this will be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(18 pts)* Bob is doing a row reduction of a 3×3 matrix, and at some stage he wants to perform the "step" below (computing the new rows from the previous rows) which he claims is a combination of row operations:

   (a) The new first row will be computed from the previous matrix as three times the second row plus four times the first row;

   (b) The new second row will be computed from the previous matrix as the third row plus two times the second row minus the first row;

   (c) The new third row will be computed from the previous matrix as the first row plus the third row.

   Is Bob correct in his assertion that this is a combination of row operations?

The "step" Bob wants to take is equivalent to left multiplying by the matrix S as indicated by

$$\underbrace{\begin{pmatrix} 4 & 3 & 0 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}}_{S} \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{matrix} 3 \textcircled{2} + 4 \textcircled{1} \\ \textcircled{3} + 2 \textcircled{2} - 1 \textcircled{1} \\ \textcircled{1} + \textcircled{3} \end{matrix}$$

We can compute

$$\det S = 1(3 \cdot 1 - 0 \cdot 2) - 0 + 1(4 \cdot 2 - 3(-1)) = 14$$

This is nonzero, so S is nonsingular, and thus a product of elementary matrices.

So Bob's step <u>is</u> a combination of row operations.

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2. *(18 pts)* The matrix A is row reduced to its reduced row echelon form R below by the matrix E below.

$$R = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad E = \begin{pmatrix} 7 & 3 & 2 \\ 4 & 2 & 2 \\ 0 & 5 & 1 \end{pmatrix}$$

(a) Find the complete set of homogeneous solutions for this matrix A. (Use $\vec{x} = (x_1, x_2, x_3, x_4)$.)

$(A|\vec{0})$ reduces to $(EA=R|E\vec{0})$, which is

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = -3x_2 - 2x_4 \\ x_3 = -4x_4 \end{array}$$

So the homogeneous solutions are $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 \\ x_2 \\ -4x_4 \\ x_4 \end{pmatrix}$

(b) Find the complete set of solutions to the system $A\vec{x} = \vec{b}$ where $\vec{b} = (1, 0, 0)$.

$(A|\vec{e}_1)$ reduces to $(EA=R|E\vec{e}_1)$, which is

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 7 \\ 0 & 0 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 7 - 3x_2 - 2x_4 \\ x_3 = 4 - 4x_4 \end{array}$$

So the homogeneous solutions are $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 - 3x_2 - 2x_4 \\ x_2 \\ 4 - 4x_4 \\ x_4 \end{pmatrix}$

(c) Find a vector $\vec{c}$ for which $A\vec{x} = \vec{c}$ has no solutions, or explain how you know that the vector $\vec{c}$ cannot exist.

We need to find a vector $\vec{c}$ for which $E\vec{c}$ has a nonzero third coordinate, since that will create a contradiction. We easily see that

$$E\vec{e}_2 = (\text{2nd column of } E) = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

So $A\vec{x} = \vec{e}_2$ has no solutions.

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3. *(18 pts)* The 3×3 nonsingular matrix B is reduced to its reduced row echelon form R by the following sequence of row operations:

(1) adding the second row to the first row;

(2) multiplying the third row by 4;

(3) switching the second and third rows.

Find elementary matrices E_1, E_2, E_3 with $B = E_3 E_2 E_1$.

The given product tells us that

$$E_1^{-1} E_2^{-1} E_3^{-1} B = I$$

and the described row reduction is represented in matrix form as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B = R = I$$

(because B is nonsingular)

Pairing as suggested by these equations, then inverting, we get

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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4. *(18 pts)* The variables a, b, c, p, q, r are related by the equations

$$\begin{aligned} a - 2b + 3c &= p \\ 2a + b - c &= q \\ 3a - 4b + 2c &= r \end{aligned}$$

We would like to increase a as quickly as possible. Is it better to hold p and q fixed and raise r, or to hold q and r fixed and raise p?

Applying Cramer's rule to $\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 3 & -4 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

we get $a = \dfrac{\det\begin{pmatrix} p & -2 & 3 \\ q & 1 & -1 \\ r & -4 & 2 \end{pmatrix}}{\det\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 3 & -4 & 2 \end{pmatrix}} = \dfrac{p(-2) - q(8) + r(-1)}{1(-2) - 2(8) + 3(-1)}$

$$a = \frac{-2p - 8q - r}{-21}$$

$$a = \frac{2p + 8q + r}{21}$$

So we have $\dfrac{\partial a}{\partial p} = \dfrac{2}{21}$, and $\dfrac{\partial a}{\partial r} = \dfrac{1}{21}$.

So we should hold q and r constant and raise p.

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5. *(12 pts)* Use the product below to find the determinant of the matrix A without either (1) multiplying any of these matrices, or (2) identifying A itself.

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&3\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}4&0&0\\0&3&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&5&0\\0&1&1\\0&3&0\end{pmatrix}\begin{pmatrix}1&0&0\\3&1&0\\3&0&1\end{pmatrix}\begin{pmatrix}&&\\&A&\\&&\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

Taking det of both sides and applying multiplicativity, we get

$$\underbrace{\det\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}}_{=1}\underbrace{\det\begin{pmatrix}1&0&3\\0&1&2\\0&0&1\end{pmatrix}}_{=1}\underbrace{\det\begin{pmatrix}4&0&0\\0&3&0\\0&0&1\end{pmatrix}}_{=12}\det\begin{pmatrix}1&5&0\\0&1&1\\0&3&0\end{pmatrix}\underbrace{\det\begin{pmatrix}1&0&0\\3&1&0\\3&0&1\end{pmatrix}}_{=1}\det A=1$$

$=1$ b/c this is two row switches

$=1$ b/c this is upper triangular

$=12$ b/c this is upper triangular

$=1$ b/c this is upper triangular

$$12\det\begin{pmatrix}1&5&0\\0&1&1\\0&3&0\end{pmatrix}\det A=1$$

For this last determinant, note that switching columns gives

$$\det\begin{pmatrix}1&5&0\\0&1&1\\0&3&0\end{pmatrix}=-\det\begin{pmatrix}1&0&5\\0&1&1\\0&0&3\end{pmatrix}=-3$$

So $-36\det A=1$, so $\det A=\frac{-1}{36}$.

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6. *(16 pts)* Prove that any list of 7 vectors in $\mathbb{R}^4$ must be linearly dependent.

Let the vectors be $\vec{a}_1, \ldots, \vec{a}_7$, with $\vec{a}_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ a_{3i} \\ a_{4i} \end{pmatrix}$.

Relations are solutions $c_1, \ldots, c_7$ to

$$c_1\vec{a}_1 + \ldots + c_7\vec{a}_7 = \vec{0}$$

Rewriting by coordinates gives

$$\begin{aligned} c_1a_{11} + c_2a_{12} + \cdots + c_7a_{17} &= 0 \\ c_1a_{21} + c_2a_{22} + \cdots + c_7a_{27} &= 0 \\ c_1a_{31} + c_2a_{32} + \cdots + c_7a_{37} &= 0 \\ c_1a_{41} + c_2a_{42} + \cdots + c_7a_{47} &= 0 \end{aligned}$$

which in matrix form is

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{17} & 0 \\ a_{21} & \vdots & \ddots & \vdots & \vdots \\ a_{31} & \vdots & & & \\ a_{41} & & \cdots & a_{47} & 0 \end{array}\right)$$

This matrix has more columns than rows, so there must be a column with no pivot and thus a free variable, so there is a significant solution and thus a significant relation between the vectors.

So the list must be linearly dependent.

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