EXAM 3

Math 216, 2019 Fall, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything near the staple – this will be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

(Scratch space. Nothing on this page will be graded!)

1. *(15 pts)* Let $M_{2\times 2}$ be the vector space of 2×2 matrices with real elements. Let $\mathcal{V}$ be the basis

$$\mathcal{V} = \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\vec{v}_2}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\vec{v}_3}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\vec{v}_4} \right\}$$

(a) $T : M_{2\times 2} \to M_{2\times 2}$ is the linear transformation $T(X) = AX$, where $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Find the matrix $[T]_{\mathcal{V}}^{\mathcal{V}}$.

$$T(\vec{v}_1) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = a\vec{v}_1 + b\vec{v}_2$$

$$T(\vec{v}_2) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} = c\vec{v}_1 + d\vec{v}_2$$

$$T(\vec{v}_3) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = a\vec{v}_3 + b\vec{v}_4$$

$$T(\vec{v}_4) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} = c\vec{v}_3 + d\vec{v}_4$$

$$[T]_{\mathcal{V}}^{\mathcal{V}} = \left([T(\vec{v}_1)]_{\mathcal{V}} \; [T(\vec{v}_2)]_{\mathcal{V}} \; [T(\vec{v}_3)]_{\mathcal{V}} \; [T(\vec{v}_4)]_{\mathcal{V}} \right) = \begin{pmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{pmatrix}$$

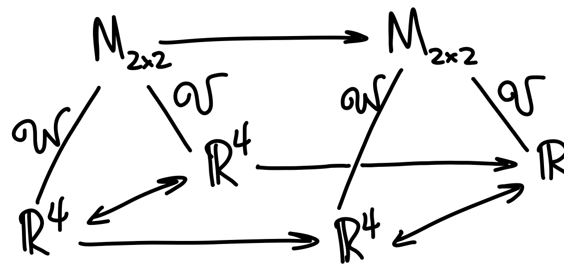
(b) Suppose $\mathcal{W} = \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}_{\vec{w}_1}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix}}_{\vec{w}_2}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 4 \end{pmatrix}}_{\vec{w}_3}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\vec{w}_4} \right\}$. Find $[I]_{\mathcal{W}}^{\mathcal{V}}$ and $[I]_{\mathcal{V}}^{\mathcal{W}}$.

$$[I]_{\mathcal{W}}^{\mathcal{V}} = \left([\vec{w}_1]_{\mathcal{V}} \; [\vec{w}_2]_{\mathcal{V}} \; [\vec{w}_3]_{\mathcal{V}} \; [\vec{w}_4]_{\mathcal{V}} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$[I]_{\mathcal{V}}^{\mathcal{W}} = \left([I]_{\mathcal{W}}^{\mathcal{V}} \right)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -3 & -4 & 1 \end{pmatrix}$$

(c) Suppose $[x]_{\mathcal{W}} = (1, 0, 2, 1)$. Compute $[T(x)]_{\mathcal{W}}$ without computing either x or $T(x)$ explicitly.

$$[T(x)]_{\mathcal{W}} = [I]_{\mathcal{V}}^{\mathcal{W}} [T]_{\mathcal{V}}^{\mathcal{V}} [I]_{\mathcal{W}}^{\mathcal{V}} [x]_{\mathcal{W}}$$



$$= [I]_{\mathcal{V}}^{\mathcal{W}} [T]_{\mathcal{V}}^{\mathcal{V}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

$$= [I]_{\mathcal{V}}^{\mathcal{W}} \begin{pmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -3 & -4 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ 2a+11c \\ 2b+11d \end{pmatrix} = \begin{pmatrix} a \\ b \\ 2a+11c \\ -10a-b-44c+11d \end{pmatrix}$$

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2. *(15 pts)* We are given the arithmetic below about the matrix A.

$$J = \underbrace{\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}}_{[I]_{\mathcal{V}}^{\mathcal{V}}} = \underbrace{\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}}_{[I]_{\mathcal{E}}^{\mathcal{V}}} \underbrace{\begin{pmatrix} & & & \\ & & A & \\ & & & \end{pmatrix}}_{[T]_{\mathcal{E}}^{\mathcal{E}}} \underbrace{\begin{array}{c} \begin{array}{cccc} [\vec{v}_1]_{\mathcal{E}} & [\vec{v}_2]_{\mathcal{E}} & [\vec{v}_3]_{\mathcal{E}} & [\vec{v}_4]_{\mathcal{E}} \end{array} \\ \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \end{array}}_{[I]_{\mathcal{V}}^{\mathcal{E}}}$$

(a) Find a Jordan basis $\mathcal{V}$ for A.

The given arithmetic is a conjugation of A to its Jordan form, so $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a Jordan basis.

(b) Find the eigenvectors and eigenvalues of A.

Interpreting from J, we have

$T(\vec{v}_1) = 3\vec{v}_1$

$T(\vec{v}_3) = 3\vec{v}_3$

$T(\vec{v}_4) = 4\vec{v}_4$

So $\vec{v}_1, \vec{v}_3, \vec{v}_4$ are eigenvectors with eigenvalues $3, 3, 4$, (resp.).

(c) Find a change of basis matrix C for which $J_2 = CJC^{-1}$ is also in Jordan form.

The rearrangement J_2 (right) comes from using the reordering $\mathcal{W} = \{\vec{v}_3, \vec{v}_1, \vec{v}_2, \vec{v}_4\}$.

$$J_2 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Then we see from below

$$\underbrace{[T]_{\mathcal{W}}^{\mathcal{W}}}_{= J_2} = [I]_{\mathcal{V}}^{\mathcal{W}} \underbrace{[T]_{\mathcal{V}}^{\mathcal{V}}}_{= J} [I]_{\mathcal{W}}^{\mathcal{V}}$$

that $C = [I]_{\mathcal{V}}^{\mathcal{W}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

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3. *(20 pts)*

(a) Show that the basis for $\mathbb{R}^3$ consisting of $\vec{v}_1 = (1, 0, 1)$, $\vec{v}_2 = (1, 2, -1)$, $\vec{v}_3 = (-1, 1, 1)$ is orthogonal.

$\vec{v}_1 \cdot \vec{v}_2 = 0$, $\vec{v}_1 \cdot \vec{v}_3 = 0$, $\vec{v}_2 \cdot \vec{v}_3 = 0$. So this basis is orthogonal.

(b) Use the basis above to find a corresponding orthonormal basis, and orthogonal matrix M.

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} / \sqrt{2}$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} / \sqrt{6}$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} / \sqrt{3}$$

$\alpha = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is orthonormal, and M below is thus orthogonal.

$$M = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$$

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(c) Find the inverse of M. M is orthogonal, so

$$M^{-1} = M^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}^T = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$

(d) Use the result of part (c) to find the inverse of the matrix A whose columns are $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$.

$A = MD$, with $D = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}$

Then $A^{-1} = D^{-1}M^{-1}$

$$= \begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{6} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/6 & 2/6 & -1/6 \\ -1/3 & 1/3 & 1/3 \end{pmatrix}$$

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4. *(16 pts)* Let V be an inner product space of photos thought of as real-valued functions on $D \subset \mathbb{R}^2$, using the L^2 inner product on D. We know that the photos f and g are elements in an orthogonal basis α for V, and the photo h is in the span of f and g.

Suppose that

$$\iint_D h(x,y)f(x,y)\,dx\,dy = 5 \quad \iint_D h(x,y)g(x,y)\,dx\,dy = 7$$

$$\iint_D f(x,y)f(x,y)\,dx\,dy = 4 \quad \iint_D g(x,y)g(x,y)\,dx\,dy = 9$$

Find the coefficients of h as a linear combination of f and g.

The given information tells us that $\|f\|=2$ and $\|g\|=3$.

So $u_1 = \frac{f}{\|f\|} = \frac{f}{2}$ and $u_2 = \frac{g}{\|g\|} = \frac{g}{3}$ form an orthonormal basis for $\text{span}\{f,g\}$.

Writing $h = c_1 u_1 + c_2 u_2$, we can find c_1, c_2 with

$$c_1 = \langle h, u_1 \rangle = \langle h, \tfrac{f}{2} \rangle = \tfrac{1}{2}\langle h,f\rangle = \tfrac{5}{2}$$

$$c_2 = \langle h, u_2 \rangle = \langle h, \tfrac{g}{3} \rangle = \tfrac{1}{3}\langle h,g\rangle = \tfrac{7}{3}$$

Then

$$\begin{aligned} h &= c_1 u_1 + c_2 u_2 \\ &= \tfrac{5}{2}\left(\tfrac{f}{2}\right) + \tfrac{7}{3}\left(\tfrac{g}{3}\right) \\ &= \tfrac{5}{4} f + \tfrac{7}{9} g \end{aligned}$$

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5. *(18 pts)* The information below is given about the functions $f, g, h, m, p \in C^\infty$. Find a list of 3 of these functions that you can show is linearly independent. *(Hint: Construct a linear transformation with values in $\mathbb{R}^3$.)*

$$
\begin{array}{rclrclrclrclrcl}
\int_0^3 f(x)\,dx &=& 3 & \int_0^3 g(x)\,dx &=& 3 & \int_0^3 h(x)\,dx &=& 6 & \int_0^3 m(x)\,dx &=& 2 & \int_0^3 p(x)\,dx &=& 1 \\
f'(1) &=& 2 & g'(1) &=& 6 & h'(1) &=& 4 & m'(1) &=& 4 & p'(1) &=& 1 \\
f(1)f(2) &=& 5 & g(1)g(2) &=& 7 & h(1)h(2) &=& 3 & m(1)m(2) &=& 4 & p(1)p(2) &=& 2 \\
\int_0^1 (f(x))^2\,dx &=& 6 & \int_0^1 (g(x))^2\,dx &=& 1 & \int_0^1 (h(x))^2\,dx &=& 8 & \int_0^1 (m(x))^2\,dx &=& 7 & \int_0^1 (p(x))^2\,dx &=& 1 \\
f'''(2) - f(3) &=& 1 & g'''(2) - g(3) &=& 3 & h'''(2) - h(3) &=& 2 & m'''(2) - m(3) &=& 2 & p'''(2) - p(3) &=& 9
\end{array}
$$

Let $T : C^\infty \to \mathbb{R}^3$ be the linear transformation

$$T(y) = \begin{pmatrix} \int_0^3 y(x)\,dx \\ y'(1) \\ y'''(2) - y(3) \end{pmatrix}$$

(from lines 1,2,5 above; lines 3,4 are not linear.) Then

$$T(f) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad T(g) = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} \quad T(h) = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} \quad T(m) = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \quad T(p) = \begin{pmatrix} 1 \\ 1 \\ 9 \end{pmatrix}$$

Notice that $T(f) \parallel T(h)$ and $T(g) \parallel T(m)$, so any trio we choose should not include more than 1 of each of these.

Let's consider then $\{f, g, p\}$. For this list, the Wronskian defined by T is

$$W = \det\Big(T(f)\ T(g)\ T(p)\Big) = \det\begin{pmatrix} 3 & 3 & 1 \\ 2 & 6 & 1 \\ 1 & 3 & 9 \end{pmatrix} = 102 \neq 0$$

So $\{f, g, p\}$ is linearly independent.

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6. *(16 pts)* Use the arithmetic given below to find a real fundamental set of solutions to the system $\vec{y}' = A\vec{y}$.

$$\begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 10 & 17 \end{pmatrix} \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} 17 & -5 \\ -10 & 3 \end{pmatrix}$$

$$[T]_{\mathcal{V}}^{\mathcal{V}} = [I]_{\mathcal{S}}^{\mathcal{V}} [T]_{\mathcal{S}}^{\mathcal{S}} [I]_{\mathcal{V}}^{\mathcal{S}}$$

So the eigenvectors are $\vec{v}_1 = \begin{pmatrix} 17 \\ -10 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$, with eigenvalues $\lambda_1 = 5$, $\lambda_2 = 7$, (resp.).

Using the theorem from class then, a f.s.s. is $\left\{ e^{5x}\begin{pmatrix} 17 \\ -10 \end{pmatrix}, e^{7x}\begin{pmatrix} -5 \\ 3 \end{pmatrix} \right\}$.

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