EXAM 2

Math 216, 2019 Fall, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything near the staple – this will be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(20 pts)*

(a) The matrices A and B are row equivalent, and we know that

$$B = \begin{pmatrix} 34 & 2 & 36 \\ 1 & 23 & 24 \\ 1 & 1 & 2 \end{pmatrix}$$

Without doing any row operations on B, find a relation between the columns of A.

Column relations are preserved by row operations:

$$\Big(A\vec{c} = \vec{0} \iff E(A\vec{c}) = \vec{0} \iff (EA)\vec{c} = \vec{0}$$

so $\vec{c}$ is a column relation for A
iff $\vec{c}$ is a column relation for EA.$\Big)$

By inspection $\vec{b}_1 + \vec{b}_2 = \vec{b}_3$, so $\vec{a}_1 + \vec{a}_2 = \vec{a}_3$.

(b) Bob knows that the list $f_1, \ldots, f_n$ is linearly dependent. But because he does not know either that these functions are analytic nor that they are solutions to any sort of differential equation, he feels that he cannot conclude that the Wronskian is identically zero, or anything else. If he is right, explain what more he would need to know to conclude that the Wronskian is identically zero; if he is wrong, what can he conclude about the Wronskian knowing no additional information?

Bob's concerns would relate to drawing conclusions <u>from</u> the Wronskian, not <u>about</u> the Wronskian.

It is a basic fact of the Wronskian that

$$\{f_1, \ldots, f_n\} \text{ l.d.} \implies W(x) = 0 \ \forall x.$$

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2. *(16 pts)* Find a fundamental set of real solutions to the constant coefficient linear differential equation $L(y) = 0$ with characteristic polynomial below.

$$p(\lambda) = (\lambda^3 - 5\lambda^2 + 19\lambda + 25)^2$$

$p(-1) = 0$, so $(\lambda+1)$ is a factor.

$$\begin{array}{r|l}
 & \lambda^2 - 6\lambda + 25 \\
\hline
\lambda+1 & \lambda^3 - 5\lambda^2 + 19\lambda + 25 \\
 & \underline{\lambda^3 + 1\lambda^2} \\
 & -6\lambda^2 + 19\lambda + 25 \\
 & \underline{-6\lambda^2 - 6\lambda} \\
 & 25\lambda + 25 \\
 & \underline{25\lambda + 25} \\
 & 0
\end{array}$$

And $\lambda^2 - 6\lambda + 25$
$= (\lambda-3)^2 + 16$
has roots $3 \pm 4i$.

So $p(\lambda) = (\lambda - (-1))^2 (\lambda - (3+4i))^2 (\lambda - (3-4i))^2$

Then a fundamental set of solutions is

$$\{e^{-x}, xe^{-x}, e^{3x}\cos 4x, e^{3x}\sin 4x, xe^{3x}\cos 4x, xe^{3x}\sin 4x\}$$

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3. *(16 pts)* Find the form of a particular solution to the equation below (do not evaluate the coefficients).

$$y'' - 6y' + 25y = \underbrace{x^2 e^{3x}\cos(4x)}_{g_1(x)} \underbrace{- 10e^{2x}\sin(5x)}_{g_2(x)}$$

$p(\lambda) = \lambda^2 - 6\lambda + 25 = (\lambda-3)^2 + 16$ has roots $3 \pm 4i$, $m=1$.

For $L(y) = g_1$, $a+bi = 3+4i$ is a root of p, $m=1$.

So
$$y_{p_1} = x(c_2x^2 + c_1x + c_0)e^{3x}\cos 4x + x(d_2x^2 + d_1x + d_0)e^{3x}\sin 4x$$

For $L(y) = g_2$, $a+bi = 2+5i$ is not a root of p.

So
$$y_{p_2} = Ae^{2x}\cos 5x + Be^{2x}\sin 5x$$

Then

$$\begin{aligned} y_p &= y_{p_1} + y_{p_2} \\ &= x(c_2x^2 + c_1x + c_0)e^{3x}\cos 4x \\ &\quad + x(d_2x^2 + d_1x + d_0)e^{3x}\sin 4x \\ &\quad + Ae^{2x}\cos 5x + Be^{2x}\sin 5x \end{aligned}$$

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4. *(16 pts)* The differential equation

$$mz'' + fz' + kz = e^{i\omega t}$$

has solution $z = (i-1)e^{i\omega t}$. Use this information to find a particular solution to

$$my'' + fy' + ky = 3\sin(\omega t)$$

$i-1 = \sqrt{2}\, e^{i(3\pi/4)}$, so $z = \sqrt{2}\, e^{i(\omega t + 3\pi/4)}$

We see

$$3\sin \omega t = 3\,\mathrm{Im}\left(e^{i\omega t}\right)$$

so

$$\begin{aligned} y &= 3\,\mathrm{Im}(z) \\ &= 3\,\mathrm{Im}\left(\sqrt{2}\, e^{i(\omega t + 3\pi/4)}\right) \\ &= 3\sqrt{2}\,\sin(\omega t + 3\pi/4). \end{aligned}$$

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5. *(16 pts)* $T : V \to W$ is a linear transformation. We define a "homogeneous solution" to be a vector $v_h \in V$ with $T(v_h) = 0$, and a "particular solution to $T(v) = w$" to be any vector $v_p \in V$ with $T(v_p) = w$.

(a) Suppose v_p is a particular solution to $T(v) = w$, and v_h is a homogenous solution. Show that $v = v_p + v_h$ is a solution to $T(v) = w$.

We need to show that $T(v_p+v_h) = W$.

$$T(v) = T(v_p+v_h) = T(v_p) + T(v_h)$$
$$= W + 0$$
$$= W.$$

So $v = v_p + v_h$ is a solution.

(b) Suppose that v is a solution and v_p is a particular solution to $T(v) = w$. Show that there is a homogeneous solution v_h for which $v = v_p + v_h$.

We need to show that $v_h = v - v_p$ is a homogeneous solution.

$$T(v_h) = T(v - v_p) = T(v) - T(v_p)$$
$$= W - W$$
$$= 0.$$

So $v = v_p + v_h$ where v_h is a homogeneous solution.

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6. *(16 pts)* We are given

$$\Big(D^5 - 5D^4 + D - 1\Big)(3x^6 e^{2x}) = g(x)$$

Use this information to find a solution to the differential equation $L(y) = g(x)$ whose characteristic polynomial is

$$p(\lambda) = (\lambda^5 - 5\lambda^4 + \lambda - 1)(\lambda - 2)^2$$

Using $L = p(D)$, we can rewrite $L(y) = g(x)$ as

$$(D^5 - 5D^4 + D - 1)(D-2)^2 y = g(x).$$

Comparing to the given, we just need to solve

$$(D-2)^2 y = 3x^6 e^{2x}$$

Recalling $(D-r)(x^k e^{rx}) = k x^{k-1} e^{rx}$ and thus guessing $y = Ax^8 e^{2x}$, we get

$$(D-2)(D-2)\, Ax^8 e^{2x} = 3x^6 e^{2x}$$

$$(D-2)\, 8Ax^7 e^{2x} = 3x^6 e^{2x}$$

$$56Ax^6 e^{2x} = 3x^6 e^{2x}$$

$$\Rightarrow 56A = 3 \Rightarrow A = \frac{3}{56} \Rightarrow y = \frac{3}{56} x^8 e^{2x}$$

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