EXAM 1

Math 216, 2019 Fall, Clark Bray.

Name: Solutions ______ Section: ______ Student ID: ______

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything near the staple – this will be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______

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1. *(18 pts)* Bob says that a complete row reduction (to its RREF, R) of the nonsingular matrix A is performed by left multiplication by the matrix G below.

$$G = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 5 \\ 0 & 2 & 9 \end{pmatrix}$$

(a) Is G actually a combination of row operations, or not?

By a cofactor expansion along the first column, we have

$$\det G = 1 \cdot (1 \cdot 9 - 2 \cdot 5) = -1$$

This is not zero, so G is nonsingular and thus is a product of elementary matrices.

(b) Find the matrix A^{-1} or show that it does not exist.

We are given that G reduces A to R, so

$$GA = R$$

We also know A is nonsingular, so $R = I$, and therefore

$$GA = I$$

It immediately follows that $A^{-1} = G$.

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2. *(18 pts)* The reduced row echelon form for A is

$$R = \begin{pmatrix} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find the complete set of solutions to $A\vec{x} = \vec{0}$.

Solving the pivot equations for the pivot variables, we get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_2 - 4x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{pmatrix}$$

(b) Suppose A times the vector $(1, 2, 3, 4)$ is $(5, 6, 7)$. Find the complete set of solutions to $A\vec{x} = (5, 6, 7)$.

Writing the given as $A\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ we see that

$(1,2,3,4)$ is a particular solution. Using also the result of part (a),

$$\vec{X} = \vec{X}_p + \vec{X}_H$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -3x_2 - 4x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{pmatrix}$$

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3. *(14 pts)* Use the arithmetic below to compute the determinant of the matrix M below *without* using cofactors or a row reduction.

$$\det\begin{pmatrix}1 & 3 & 2\\ 2 & 2 & 1\\ 3 & 1 & 1\end{pmatrix} = -4, \quad \det\begin{pmatrix}1 & 1 & 2\\ 2 & 1 & 1\\ 3 & 1 & 1\end{pmatrix} = -1, \quad M = \begin{pmatrix}1 & 6 & 2\\ 2 & 5 & 1\\ 3 & 4 & 1\end{pmatrix}$$

The matrices above all have the same first and third columns. On the second columns we note easily that

$$\begin{pmatrix}6\\5\\4\end{pmatrix} = 1\begin{pmatrix}3\\2\\1\end{pmatrix} + 3\begin{pmatrix}1\\1\\1\end{pmatrix}$$

so by multilinearity we have

$$\det\begin{pmatrix}1 & 6 & 2\\ 2 & 5 & 1\\ 3 & 4 & 1\end{pmatrix} = 1\det\begin{pmatrix}1 & 3 & 2\\ 2 & 2 & 1\\ 3 & 1 & 1\end{pmatrix} + 3\det\begin{pmatrix}1 & 1 & 2\\ 2 & 1 & 1\\ 3 & 1 & 1\end{pmatrix}$$

$$\det M = 1(-4) + 3(-1)$$

$$= -7$$

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4. *(20 pts)* Bob is hoping to be able to use vector space theorems to study L, the set of all positive-real-valued functions defined on $\mathbb{R}$, with operations $\oplus$ and $\otimes$ defined by

$$\begin{aligned} (f \oplus g)(x) &= f(x)g(x) \\ (c \otimes f)(x) &= |c|f(x) \end{aligned}$$

(a) Is the condition below satisfied? Prove or find a counterexample

$$u + (v + w) = (u + v) + w \text{ for all vectors } u, v, w$$

$$\big(f \oplus (g \oplus h)\big)(x) = f(x)\,(g \oplus h)(x) = f(x)\,g(x)\,h(x)$$

$$\big((f \oplus g) \oplus h\big)(x) = (f \oplus g)(x)\,h(x) = f(x)\,g(x)\,h(x)$$

These are equal so the condition is satisfied.

(b) Is L a vector space? Prove or show a condition that is not satisfied.

One condition that fails is

$$(c+d)v = cv + dv$$

An example of the failure is shown by

$$(-1+1) \otimes e^x = 0\,e^x = 0$$

$$-1 \otimes e^x \oplus 1 \otimes e^x = e^x \oplus e^x = e^{2x}$$

which would need to be equal, but are not.

(Another failed condition is $c(u+v) = cu + cv$; let $c=2$, $u=v=e^x$ ...)

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5. *(15 pts)* Show that the plane $3x - 2y + z = 0$ in $\mathbb{R}^3$ is a vector space.

We will show that this plane P is a subspace of $\mathbb{R}^3$.

Closed under addition:

Suppose $\vec{x}_1 = (x_1, y_1, z_1)$, $\vec{x}_2 = (x_2, y_2, z_2) \in P$.

Then

$$3x_1 - 2y_1 + z_1 = 0, \quad 3x_2 - 2y_2 + z_2 = 0$$

Adding these equations we get

$$3(x_1 + x_2) - 2(y_1 + y_2) + (z_1 + z_2) = 0$$

which shows that $\vec{x}_1 + \vec{x}_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in P$

Closed under scalar multiplication:

Suppose $\vec{x} = (x, y, z) \in P$.

Then

$$3x - 2y + z = 0$$

Multiplying both sides by c gives us

$$3(cx) - 2(cy) + (cz) = 0$$

which shows that $c\vec{x} = (cx, cy, cz) \in P$.

So P is a subspace of $\mathbb{R}^3$ and thus a vector space.

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6. *(15 pts)* Suppose that p_1, p_2, p_3, p_4 are all quadratic (at most) polynomials. Without using the notion of dimension, show that this list must be linearly dependent.

Let $p_i = a_i x^2 + b_i x + c_i$ for $i = 1,2,3,4$. We consider relations

$$k_1 p_1 + k_2 p_2 + k_3 p_3 + k_4 p_4 = 0$$

Collecting terms, we have

$$\begin{aligned} &(a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4)x^2 & & 0x^2 \\ +&(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4)x & = & +0x \\ +&(c_1k_1 + c_2k_2 + c_3k_3 + c_4k_4) & & +0 \end{aligned}$$

and thus the system

$$\begin{aligned} a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4 &= 0 \\ b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4 &= 0 \\ c_1k_1 + c_2k_2 + c_3k_3 + c_4k_4 &= 0 \end{aligned}$$

with augmented matrix

$$\left(\begin{array}{cccc|c} a_1 & a_2 & a_3 & a_4 & 0 \\ b_1 & b_2 & b_3 & b_4 & 0 \\ c_1 & c_2 & c_3 & c_4 & 0 \end{array}\right)$$

With at most 1 pivot in each of 3 rows, there must be one of the 4 columns with no pivot, so there is a free variable, and thus in addition to the trivial solution there is also a nontrivial solution.

Thus we have a significant relation among p_1, p_2, p_3, p_4, so the list is linearly dependent.

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