EXAM 3

Math 216, 2019 Spring, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything on the QR codes or nearby print, or near the staple.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(20 pts)* The function $T : P_2 \to P_3$ is a linear transformation, and $T(f)$ is defined as the unique antiderivative F of f with $F(0) = 0$. Let $\mathcal{S}_2 = \{1, x, x^2\}$, $\mathcal{S}_3 = \{1, x, x^2, x^3\}$, $\mathcal{V} = \{1+x, x+x^2, x^2\}$, $\mathcal{W} = \{1 - x + 3x^2, x + x^2 - x^3, x^2 - 4x^3, 4x^3\}$.

(a) Compute $M = [T]_{\mathcal{S}_2}^{\mathcal{S}_3}$.

$$[T]_{\mathcal{S}_2}^{\mathcal{S}_3} = \begin{pmatrix} [T(1)]_{\mathcal{S}_3} & [T(x)]_{\mathcal{S}_3} & [T(x^2)]_{\mathcal{S}_3} \end{pmatrix} = \begin{pmatrix} [x]_{\mathcal{S}_3} & [\tfrac{1}{2}x^2]_{\mathcal{S}_3} & [\tfrac{1}{3}x^3]_{\mathcal{S}_3} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

(b) Compute $[I]_{\mathcal{V}}^{\mathcal{S}_2}$ and $[I]_{\mathcal{W}}^{\mathcal{S}_3}$.

$$[I]_{\mathcal{V}}^{\mathcal{S}_2} = \begin{pmatrix} [1+x]_{\mathcal{S}_2} & [x+x^2]_{\mathcal{S}_2} & [x^2]_{\mathcal{S}_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$[I]_{\mathcal{W}}^{\mathcal{S}_3} = \begin{pmatrix} [1-x+3x^2]_{\mathcal{S}_3} & [x+x^2-x^3]_{\mathcal{S}_3} & [x^2-4x^3]_{\mathcal{S}_3} & [4x^3]_{\mathcal{S}_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & -1 & -4 & 4 \end{pmatrix}$$

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(c) The matrix $[T]_{\mathcal{V}}^{\mathcal{W}}$ can be written as QMR. Find either Q or Q^{-1} (whichever you prefer), and find either R or R^{-1} (whichever you prefer).

$$
\begin{aligned}
[T]_{\mathcal{V}}^{\mathcal{W}} &= [I]_{\mathcal{S}_3}^{\mathcal{W}} \, [T]_{\mathcal{S}_2}^{\mathcal{S}_3} \, [I]_{\mathcal{V}}^{\mathcal{S}_2} \\
&= \left([I]_{\mathcal{W}}^{\mathcal{S}_3}\right)^{-1} \; M \; [I]_{\mathcal{V}}^{\mathcal{S}_2} \\
&= Q \quad M \quad R
\end{aligned}
$$

So $Q^{-1} = [I]_{\mathcal{W}}^{\mathcal{S}_3}$ (see prev. page), $R = [I]_{\mathcal{V}}^{\mathcal{S}_2}$ (see prev. page).

(d) Suppose we know that $[T(f)]_{\mathcal{W}} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}$. Compute $[T]_{\mathcal{S}_2}^{\mathcal{S}_3}[f]_{\mathcal{S}_2}$ *without* finding f explicitly.

$$
[T]_{\mathcal{S}_2}^{\mathcal{S}_3} [f]_{\mathcal{S}_2} = [T(f)]_{\mathcal{S}_3} = [I]_{\mathcal{W}}^{\mathcal{S}_3} [T(f)]_{\mathcal{W}}
$$

by definition of $[T]_{\mathcal{S}_2}^{\mathcal{S}_3}$.

immediate application of the change of basis matrix.

$$
= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & -1 & -4 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 \\ 2 \\ 3 \\ -2 \end{pmatrix}
$$

$$
\begin{tikzcd}
P_2 \arrow[rr, "T"] \arrow[dr, leftrightarrow, "\mathcal{S}_2"] \arrow[dd, leftrightarrow, "\mathcal{V}"'] & & P_3 \arrow[dr, leftrightarrow, "\mathcal{S}_3"] \arrow[dd, leftrightarrow, "\mathcal{W}"'] & \\
& \mathbb{R}^3 \arrow[rr, "{[T]_{\mathcal{S}_2}^{\mathcal{S}_3}}"] & & \mathbb{R}^4 \\
\mathbb{R}^3 \arrow[rr, "{[T]_{\mathcal{V}}^{\mathcal{W}}}"] \arrow[ur, leftrightarrow] & & \mathbb{R}^4 \arrow[ur, leftrightarrow] &
\end{tikzcd}
$$

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2. *(15 pts)* Find the diagonalization (the resulting diagonal matrix AND the basis used to achieve it) of the matrix A below.

$$A = \begin{pmatrix} 8 & -10 \\ 3 & -3 \end{pmatrix}$$

$$p(\lambda) = (8-\lambda)(-3-\lambda) + 30 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$$

So the eigenvectors are 2 and 3.

$\underline{\lambda=2}$: $A - \lambda I = \begin{pmatrix} 6 & -10 \\ 3 & -5 \end{pmatrix}$

rank $=1 \Rightarrow \dim(NS) = 1$

So $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ is the only eigenvector.

$\underline{\lambda=3}$: $A - \lambda I = \begin{pmatrix} 5 & -10 \\ 3 & -6 \end{pmatrix}$

rank $=1 \Rightarrow \dim(NS) = 1$

So $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is the only eigenvector.

Then $\gamma = \left\{ \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ gives $[T]_\gamma^\gamma = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

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3. *(15 pts)* For v, w, x in the inner product space V, we are given the following information:

$$\begin{pmatrix} \langle v, v\rangle & \langle v, w\rangle & \langle v, x\rangle \\ & \langle w, w\rangle & \langle w, x\rangle \\ & & \langle x, x\rangle \end{pmatrix} = \begin{pmatrix} 5 & 2 & 3 \\ & 5 & 3 \\ & & 3 \end{pmatrix}$$

Compute the angle between $v + w$ and x in V.

$$\|v+w\|^2 = \langle v+w, v+w\rangle = \langle v,v\rangle + 2\langle v,w\rangle + \langle w,w\rangle = 14$$

So $\|v+w\| = \sqrt{14}$.

$$\|x\| = \sqrt{\langle x,x\rangle} = \sqrt{3}.$$

$$\langle v+w, x\rangle = \langle v,x\rangle + \langle w,x\rangle = 6$$

Then the angle is

$$\theta = \arccos\left(\frac{\langle v+w, x\rangle}{\|v+w\| \, \|x\|}\right) = \arccos\left(\frac{6}{\sqrt{14}\,\sqrt{3}}\right)$$

$$= \arccos\left(\sqrt{\frac{6}{7}}\right)$$

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4. *(15 pts)* The basis $\mathcal{V}$ for $\mathbb{R}^3$ consists of the three vectors below.

$$\vec{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}$$

Find the coordinates of $\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ with respect to $\mathcal{V}$.

The above vectors are pairwise orthogonal and have unit magnitude, so $\mathcal{V}$ is an orthonormal basis.

So the coordinates are computed as projections.

$$[\vec{x}]_{\mathcal{V}} = \begin{pmatrix} \vec{x}\cdot\vec{v}_1 \\ \vec{x}\cdot\vec{v}_2 \\ \vec{x}\cdot\vec{v}_3 \end{pmatrix} = \begin{pmatrix} 5/\sqrt{2} \\ 9/\sqrt{6} \\ -3/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 5/\sqrt{2} \\ 3\sqrt{3}/2 \\ -\sqrt{3} \end{pmatrix}$$

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5. *(20 pts)* Find a fundamental set of real solutions to the system $\vec{y}' = M\vec{y}$.

$$M = \begin{pmatrix} 2 & -2 \\ 5 & 0 \end{pmatrix}$$

$$p(\lambda) = (2-\lambda)(-\lambda) + 10 = \lambda^2 - 2\lambda + 10 = (\lambda-1)^2 + 9$$

So the eigenvalues are $1 \pm 3i$.

For $\lambda = 1+3i$, we have $A - \lambda I = \begin{pmatrix} 1-3i & -2 \\ 5 & -1-3i \end{pmatrix}$.

rank $=1 \Rightarrow \dim(NS) = 1 \Rightarrow \begin{pmatrix} 2 \\ 1-3i \end{pmatrix}$ is the eigenvector.

So a solution is $e^{(1+3i)x}\begin{pmatrix} 2 \\ 1-3i \end{pmatrix}$ (and its conjugate too).

$$e^{(1+3i)x}\begin{pmatrix} 2 \\ 1-3i \end{pmatrix} = e^x(\cos 3x + i\sin 3x)\begin{pmatrix} 2 \\ 1-3i \end{pmatrix}$$

$$= e^x\begin{pmatrix} 2\cos 3x \\ \cos 3x + 3\sin 3x \end{pmatrix} + i\, e^x \begin{pmatrix} 2\sin 3x \\ \sin 3x - 3\cos 3x \end{pmatrix}$$

So a real f.s.s. is

$$\left\{ e^x\begin{pmatrix} 2\cos 3x \\ \cos 3x + 3\sin 3x \end{pmatrix}, \; e^x \begin{pmatrix} 2\sin 3x \\ \sin 3x - 3\cos 3x \end{pmatrix} \right\}$$

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6. *(15 pts)* Find a fundamental set of real solutions to the system $\vec{y}' = B\vec{y}$.

$$B = \begin{pmatrix} 6 & 9 \\ -1 & 0 \end{pmatrix} = [T]_{\mathcal{S}}^{\mathcal{S}}$$

(Hint: $(3,-1)$ is the only eigenvector. You may find a convenient use for the vector $(1,0)$.)

Choose the new basis $\mathcal{V} = \left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. Then

$$[T]_{\mathcal{V}}^{\mathcal{V}} = [I]_{\mathcal{S}}^{\mathcal{V}} [T]_{\mathcal{S}}^{\mathcal{S}} [I]_{\mathcal{V}}^{\mathcal{S}} = P^{-1} B P$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 6 & 9 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} = K$$

$\vec{y}' = B\vec{y}$ then becomes $\vec{y}' = PKP^{-1}\vec{y}$, $\vec{z}' = K\vec{z}$ with $\vec{y} = P\vec{z}$.

$$z_1' = 3z_1 + z_2$$
$$z_2' = 3z_2$$

Backsolving: $z_2 = c_2 e^{3x}$, so 1st equation becomes

$z_1' = 3z_1 + c_2 e^{3x}$ Homog: $z_{1H} = c_1 e^{3x}$

Partic.: $p(\lambda) = \lambda - 3$ $r = a+bi = 3$ root w/ $m=1$, so $z_{1p} = Axe^{3x}$.

$(Axe^{3x})' = 3(Axe^{3x}) + c_2 e^{3x}$

$\shortparallel$

$Ae^{3x} + 3Axe^{3x}$ $\Rightarrow A = c_2$

Then $z_1 = c_1 e^{3x} + c_2 x e^{3x}$.

So $\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{3x} + c_2 x e^{3x} \\ c_2 e^{3x} \end{pmatrix}$

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$$= c_1 e^{3x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{3x} \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

So a f.s.s. for $\vec{z}$ is

$$\left\{ e^{3x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e^{3x} \begin{pmatrix} x \\ 1 \end{pmatrix} \right\}.$$

With $\vec{y} = P\vec{z}$ we then have a f.s.s. for $\vec{y}$ is

$$\left\{ e^{3x} \begin{pmatrix} 3 \\ -1 \end{pmatrix}, e^{3x} \begin{pmatrix} 3x+1 \\ -x \end{pmatrix} \right\}$$

Alt: Noting that $\mathcal{V}$ is a Jordan basis, we write a f.s.s.

$$e^{xA}\vec{v}_1 = e^{3x}(\vec{v}_1) = e^{3x} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = e^{3x} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$e^{xA}\vec{v}_2 = e^{3x}(\vec{v}_2 + x\vec{v}_1) = e^{3x}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + x\begin{pmatrix} 3 \\ -1 \end{pmatrix}\right) = e^{3x} \begin{pmatrix} 3x+1 \\ -x \end{pmatrix}$$

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