EXAM 2

Math 216, 2019 Spring, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything on the QR codes or nearby print, or near the staple.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(20 pts)* The matrix A with columns $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_4$ has reduced row echelon form R below.

$$R = \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find a relation among the vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_4$. $\vec{v}_1$ $\vec{v}_2$ $\vec{v}_3$ $\vec{v}_4$

We directly see that $4\vec{v}_1 + 5\vec{v}_2 - \vec{v}_4 = \vec{0}$.

Column relations are preserved by row operations, so we also have

$$4\vec{a}_1 + 5\vec{a}_2 - \vec{a}_4 = \vec{0}$$

(b) Find a basis for the row space of A.

The pivot rows of R give us such a basis:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 5 \end{pmatrix} \right\}$$

(c) Find a basis for the column space of A.

The pivot columns of A give us such a basis:

$$\{ \vec{a}_1, \vec{a}_2 \}$$

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2. *(10 pts)* The functions f_1, f_2, f_3 are given below. Decide if this trio is linearly independent.

$$
\begin{aligned}
f_1(x) &= 3\sin(x) + 1\cos(x) + 2\cos\left(x - \frac{\pi}{3}\right) \\
f_2(x) &= 1\sin(x) + 4\cos(x) + 1\cos\left(x - \frac{\pi}{3}\right) \\
f_3(x) &= 2\sin(x) + 2\cos(x) + 5\cos\left(x - \frac{\pi}{3}\right)
\end{aligned}
$$

The angle addition formula gives us

$$\cos\left(x - \frac{\pi}{3}\right) = \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x$$

So f_1, f_2, f_3 are 3 vectors in the 2-dimensional $\operatorname{span}\{\sin x, \cos x\}$.

So this list is linearly dependent.

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3. *(20 pts)* Your friend Bob asks you to help him find all possible solutions to the differential equation below on the interval $(0, \infty)$, and he knows somehow that the three functions x, x^2, and x^3 are each related to the answer.

$$x^2y'' - 4xy' + 6y = 10x$$

(a) Bob's first thought is to try to find roots of the characteristic polynomial. Is this a good way to start? Explain.

This is not a constant coefficient equation, so there is no such thing!

(b) Identify a relevant linear differential operator L and apply it to each of the three functions given above.

$L(y) = x^2y'' - 4xy' + 6y$ $\quad$ (Alt: $L = x^2D^2 - 4xD + 6$)

$L(x) = x^2(0) - 4x(1) + 6(x) = 2x$

$L(x^2) = x^2(2) - 4x(2x) + 6(x^2) = 0$

$L(x^3) = x^2(6x) - 4x(3x^2) + 6(x^3) = 0$

(c) Use the results of the previous part to find the general solution.

$L(x) = 2x \implies L(5x) = 10x \implies y_p = 5x$ is a particular solution.

$\left.\begin{matrix} L(x^2) = 0 \\ L(x^3) = 0 \end{matrix}\right\} \implies x^2, x^3$ are homogenous solutions.

2nd order, cont. coeff. fns, $g_n(x) \neq 0$ on domain $\implies$ hom. sol. set is a 2-dim vector space

x^2, x^3 are l.i. and thus a basis in hom. sol. set.

So the general solution is $y = 5x + c_1x^2 + c_2x^3$.

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4. *(20 pts)* Find a fundamental set of real solutions to the constant coefficient linear differential equation whose characteristic polynomial is below.

$$p(\lambda) = (\lambda - 4)^3(\lambda^2 + 2\lambda + 26)^2$$

$\lambda^2 + 2\lambda + 26 = (\lambda+1)^2 + 25 = 0 \implies \lambda = -1 \pm 5i$.

Then p factors as

$$p(\lambda) = (\lambda - 4)^3 (\lambda - (-1+5i))^2 (\lambda - (-1-5i))^2$$

By the theorem from class, we have the real fundamental set

$$\{e^{4x}, xe^{4x}, x^2e^{4x}, e^{-x}\cos 5x, e^{-x}\sin 5x, xe^{-x}\cos 5x, xe^{-x}\sin 5x\}$$

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5. *(10 pts)* Find all of the cube roots (in rectangular form) of $8i$.

One cube root of $\alpha = 8e^{i(-3\pi/2)}$ is $r = 2e^{i(-\pi/2)} = -2i$.

The cube roots of unity are $1, \rho, \rho^2$ with $\rho = e^{i(2\pi/3)}$.

The three cube roots of α then are

$$r = 2e^{i(-\pi/2)} \cdot 1 = -2i$$

$$r\rho = 2e^{i(-\pi/2)} \cdot e^{i(2\pi/3)} = 2e^{i(\pi/6)} = \sqrt{3} + i$$

$$r\rho^2 = 2e^{i(-\pi/2)} \cdot e^{i(4\pi/3)} = 2e^{i(5\pi/6)} = -\sqrt{3} + i$$

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6. *(20 pts)* Prove that the kernel of a linear transformation $T : V \to W$ must be a subspace of V.

We will show that $\ker(T)$ is closed under (1) addition and (2) scalar multiplication.

(1) Suppose $x, y \in \ker(T)$.

Then $T(x) = 0$, $T(y) = 0$.

$\Rightarrow T(x+y) = T(x) + T(y) = 0.$

So we must have $x + y \in \ker(T)$.

Thus $\ker(T)$ is closed under addition.

(2) Suppose $x \in \ker(T)$.

Then $T(x) = 0$.

$\Rightarrow T(cx) = cT(x) = 0.$

So we must have $cx \in \ker(T)$.

Thus $\ker(T)$ is closed under scalar multiplication.

So $\ker(T)$ is a subspace of V.

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