# EXAM 1

Math 216, 2019 Spring, Clark Bray.

Name: Solutions Section:______ Student ID:____________

## GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

## WRITING RULES

Do not write anything on the QR codes or nearby print, or near the staple.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

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## DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(18 pts)* Bob correctly applies elementary row operations to the augmented matrix $(A|\vec{b})$ resulting in the matrix below.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(a) Find the complete set of solutions to the system $A\vec{x} = \vec{b}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} ①-② \\ ② \\ ③ \end{array} \Rightarrow \begin{array}{l} x = -1+2z \\ y = 3-2z \end{array}$$

Then the complete set of solutions is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1+2z \\ 3-2z \\ z \end{pmatrix}$$

(b) Bob states correctly that the applications of elementary row operations he performed to result in the above is equivalent to multiplication by the matrix $E$ below.

$$E = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 3 & 3 \\ 0 & 1 & 0 \end{pmatrix}$$

He further says that he has found a vector $\vec{c}$ for which $A\vec{x} = \vec{c}$ has no solutions. If he is right, find such a vector $\vec{c}$; if he is wrong, explain how you know.

Multiplying $(A|\vec{c})$ by $E$ gives us $\left(\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 2 & E\vec{c} \\ 0 & 0 & 0 & \end{array}\right)$

where we can see we would have a contradiction when

$$E\vec{c} = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 3 & 3 \\ 0 & 1 & 0 \end{pmatrix} \vec{c} = \begin{pmatrix} ? \\ ? \\ 1 \end{pmatrix}$$

This is the case with $\vec{c} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, for example.

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2. *(14 pts)* Bob is playing with the matrices $A$ and $B$ below.

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

He writes: "Direct computation shows $AB = AAA$. Then

$$\begin{aligned} AB &= AAA \\ ABA &= AAAA \\ BA &= AAA \end{aligned}$$

which implies $AB = BA$... but direct computation shows this is wrong! Help!".

Identify the error in Bob's reasoning to help him resolve this seeming contradiction. Be as clear as you can about where his reasoning is invalid, and why.

In this step, Bob left cancels, which is only valid if the cancelled matrix has the uniqueness property.

But $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ has $\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ does not have a pivot in every column, and thus does not have this required property.

So this left cancellation is invalid.

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3. *(14 pts)* Interpret the matrix $E$ below as a row operation, and use that interpretation to find $E^{-1}$, explaining your reasoning of course. (DO NOT use the usual row reduction algorithm or a memorized formula for inverses of elementary matrices).

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

E performs the row operation $R_1$ that adds 3 times the fifth row to the second row.

This operation is clearly reversed by $R_2$ that subtracts 3 times the fifth row from the second row.

So

$$I \xrightarrow{R_1} \quad \xrightarrow{R_2} I$$

and thus

$$FEI = I$$

$$\Rightarrow FE = I$$

So $E^{-1} = F$, the matrix that performs $R_2$:

$$E^{-1} = F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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4. *(18 pts)*

(a) The surface $S$ parametrized by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3\rho\sin\phi\cos\theta - 2\rho\sin\phi\sin\theta \\ \rho\sin\phi\sin\theta + 2\rho\cos\phi \\ \rho\sin\phi\cos\theta + \rho\cos\phi \end{pmatrix}$$

can be viewed as the image of the sphere parametrization $(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$ by way of a convenient function. Use this point of view to compute the volume contained inside of $S$.

The above is $\underbrace{\begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi \end{pmatrix}$

So $S$ is the image of the sphere of radius $\rho$ by

$$T(\vec{x}) = A\vec{x}$$

which stretches volumes by $|\det A| = |-1| = 1$.

So the volume is $(1)(\frac{4}{3}\pi\rho^3) = \frac{4}{3}\pi\rho^3$.

(b) The $3 \times 3$ matrix $M$ has rank equal to 2, and is reduced to its reduced row echelon form by (1) adding the third row to the first row, then (2) multiplying the second row by 4, then (3) adding 4 times the second row to the third row, then (4) subtracting 5 times the third row from the first row. Compute the determinant of $M$.

Let $\text{rref}(M) = R$. The above information tells us that

$$(\det M)(1)(4)(1)(1) = \det R.$$

Because $\text{rank}(M) = 2$, $R$ must have a row of zeroes, so $\det R = 0$. Thus $\det M = 0$.

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5. *(18 pts)*

(a) The $3 \times 3$ matrix $A$ has columns $\vec{v}_1 - 2\vec{v}_2$, $\vec{v}_2 - 5\vec{v}_3$, $\vec{v}_1 + 3\vec{v}_2 + \vec{v}_3$. The $3 \times 3$ matrix $B$ has columns $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ and is known to have determinant equal to 6. Compute the determinant of $A$.

$$A = \begin{pmatrix} | & | & | \\ \vec{v}_1 - 2\vec{v}_2 & \vec{v}_2 - 5\vec{v}_3 & \vec{v}_1 + 3\vec{v}_2 + \vec{v}_3 \\ | & | & | \end{pmatrix} = \underbrace{\begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix}}_{B} \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 3 \\ 0 & -5 & 1 \end{pmatrix}$$

Then

$$\det A = \det B \det \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 3 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= (6)(26) = 156$$

(b) Use the two determinants given below to compute the determinant of the matrix $D$ given below.

$$\det \overbrace{\begin{pmatrix} 1 & 7 & 8 \\ 3 & 2 & 2 \\ 2 & 3 & 5 \end{pmatrix}}^{A} = -33 \qquad \det \overbrace{\begin{pmatrix} 1 & 1 & 8 \\ 3 & 2 & 2 \\ 2 & 1 & 5 \end{pmatrix}}^{B} = -11 \qquad D = \begin{pmatrix} 1 & 8 & 8 \\ 3 & 4 & 2 \\ 2 & 4 & 5 \end{pmatrix}$$

We use multilinearity in the second column, and the observation that

$$1\begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix}$$

to conclude

$$1 \cdot \det A + 1 \cdot \det B = \det D$$

$$1 \cdot (-33) + 1 \cdot (-11) = \det D = -44$$

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6. *(18 pts)*

(a) What is the *definition* of linear dependence as applied to the list of vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^m$?

The list $\vec{v}_1, \ldots, \vec{v}_n$ is linearly dependent if there are $c_1, \ldots, c_n$, not all zero, with

$$c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = \vec{0}.$$

(b) Decide if the list of vectors $\vec{v}_1 = (4, 1, 2)$, $\vec{v}_2 = (1, 2, 1)$, $\vec{v}_3 = (5, 1, 0)$ is linearly dependent or linearly independent.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$$\iff \quad c_1\begin{pmatrix}4\\1\\2\end{pmatrix} + c_2\begin{pmatrix}1\\2\\1\end{pmatrix} + c_3\begin{pmatrix}5\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\iff \quad \begin{aligned} 4c_1 + 1c_2 + 5c_3 &= 0 \\ 1c_1 + 2c_2 + 1c_3 &= 0 \\ 2c_2 + 1c_2 + 0c_3 &= 0 \end{aligned}$$

$$\iff \quad \underbrace{\begin{pmatrix}4 & 1 & 5\\1 & 2 & 1\\2 & 1 & 0\end{pmatrix}}_{A}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$\det A = -17 \neq 0$, so the trivial solution $\vec{c} = \vec{0}$ is the unique solution to the matrix equation, and thus also the original vector equation. So the list is linearly independent.

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