EXAM 3

Math 216, 2018 Fall, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators. Scratch paper is allowed, but (1) it must be from the instructor, (2) it must be returned with the exam, and (3) it will NOT be graded.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything on the QR codes or near the staple.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can ONLY be done on the front or back of the page the question is written on.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

(Nothing on this page will be graded!)

1. *(20 pts)* The pair $\mathcal{F} = \{f_1, f_2\}$, with $f_1 = e^x + e^{-x}$ and $f_2 = e^x - e^{-x}$, is a basis for the vector space V. Another basis for V is the pair $\mathcal{G} = \{g_1, g_2\}$, with $g_1 = e^x$ and $g_2 = e^{-x}$. The linear transformation $D : V \to V$ is defined by $D(u) = u'$.

(a) Find the change of basis matrix $[I]_{\mathcal{F}}^{\mathcal{G}}$.

$$f_1 = 1 g_1 + 1 g_2 \Rightarrow [f_1]_G = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$f_2 = 1 g_1 + (-1) g_2 \Rightarrow [f_2]_G = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$[I]_{\mathcal{F}}^{G} = \Big[[f_1]_G \ [f_2]_G \Big] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(b) Find the matrix $[D]_{\mathcal{F}}^{\mathcal{F}}$ without explicitly computing the derivatives of f_1 and f_2.

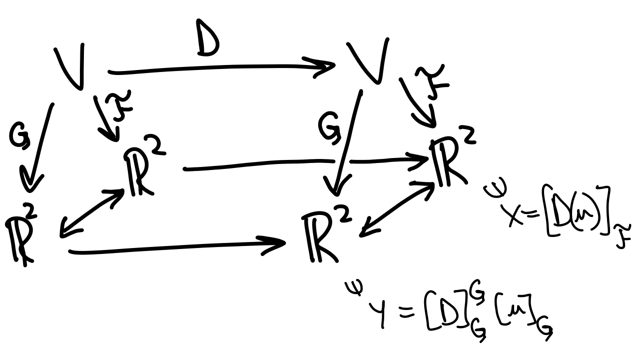
$$D(g_1) = g_1' = e^x = 1 g_1 + 0 g_2 = [D(g_1)]_G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$D(g_2) = g_2' = -e^{-x} = 0 g_1 + (-1) g_2 \Rightarrow [D(g_2)]_G = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$[D]_G^G = \Big[[D(g_1)]_G \ [D(g_2)]_G \Big] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[D]_{\mathcal{F}}^{\mathcal{F}} = [I]_G^{\mathcal{F}} [D]_G^G [I]_{\mathcal{F}}^G = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(c) What is the relationship in general between $x = [D(u)]_{\mathcal{F}}$ and $y = [D]_{\mathcal{G}}^{\mathcal{G}}[u]_{\mathcal{G}}$?



$$y = [I]_{\mathcal{F}}^{G} \, x$$

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2. *(15 pts)* Find the diagonal matrix E and the invertible matrix M such that

$$M^{-1}EM = \begin{pmatrix} 7 & -2 \\ 15 & -4 \end{pmatrix}$$

$$p(\lambda) = (7-\lambda)(-4-\lambda) - (15)(-2) = -28 - 3\lambda + \lambda^2 + 30$$
$$= \lambda^2 - 3\lambda + 2$$
$$= (\lambda-1)(\lambda-2) \Rightarrow \text{evals are } 1, 2.$$

For $\lambda = 1$:

$A - \lambda I = \begin{pmatrix} 6 & -2 \\ 15 & -5 \end{pmatrix}$ has rank $=1 \Rightarrow \dim(NS) = 1$, so $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is the only eigenvector.

For $\lambda = 2$:

$A - \lambda I = \begin{pmatrix} 5 & -2 \\ 15 & -6 \end{pmatrix}$ has rank $=1 \Rightarrow \dim(NS) = 1$, so $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is the only eigenvector.

Then with $\mathcal{V} = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix} \right\}$,

$$\begin{pmatrix} 7 & -2 \\ 15 & -4 \end{pmatrix} = [I]_{\mathcal{V}}^{\mathcal{S}} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} [I]_{\mathcal{S}}^{\mathcal{V}}$$
$$= \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}_{=E} \underbrace{\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}}_{=M}$$

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3. *(15 pts)* Find the angle between the vectors $f(x) = 1$ and $g(x) = x$ in the inner product space $C[0,1]$ with the inner product defined by $\langle v, w\rangle = \int_0^1 xv(x)w(x)\,dx$.

$$\|f\|^2 = \int_0^1 x \cdot 1 \cdot 1 \, dx = \int_0^1 x \, dx = \frac{1}{2} \quad \Rightarrow \|f\| = \sqrt{1/2}$$

$$\|g\|^2 = \int_0^1 x \cdot x \cdot x \, dx = \int_0^1 x^3 \, dx = \frac{1}{4} \quad \Rightarrow \|g\| = \frac{1}{2}$$

$$\langle f, g\rangle = \int_0^1 x \cdot 1 \cdot x \, dx = \int_0^1 x^2 \, dx = \frac{1}{3}$$

$$\theta = \arccos\left(\frac{\langle f, g\rangle}{\|f\|\,\|g\|}\right)$$

$$= \arccos\left(\frac{1/3}{\sqrt{1/2} \cdot 1/2}\right)$$

$$= \arccos\left(\frac{2\sqrt{2}}{3}\right)$$

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4. *(15 pts)* Note that in the inner product space $C[0, 2\pi]$ with the usual inner product, the vectors $\cos x$ and $\sin x$ are orthogonal and each has magnitude $\sqrt{\pi}$.

Suppose that a vector $f(x)$ is known to be a linear combination of $\cos x$ and $\sin x$. Find a formula to compute the coefficients a and b for which $f(x) = a\cos x + b\sin x$. (Be sure to explain your reasoning.)

The given observations tell us that

$$\mathcal{U} = \left\{ \frac{1}{\sqrt{\pi}} \cos x \, , \, \frac{1}{\sqrt{\pi}} \sin x \right\}$$

is an orthonormal basis for $V = \operatorname{span}\{\cos x, \sin x\}$, with coordinates therefore equal to projections:

$$[f]_{\mathcal{U}} = \begin{pmatrix} \langle f, \frac{1}{\sqrt{\pi}} \cos x \rangle \\ \langle f, \frac{1}{\sqrt{\pi}} \sin x \rangle \end{pmatrix} = \begin{pmatrix} \int_0^{2\pi} f(x) \frac{1}{\sqrt{\pi}} \cos x \, dx \\ \int_0^{2\pi} f(x) \frac{1}{\sqrt{\pi}} \sin x \, dx \end{pmatrix}$$

So

$$f = \left(\int_0^{2\pi} f(x) \frac{1}{\sqrt{\pi}} \cos x \, dx \right) \left(\frac{1}{\sqrt{\pi}} \cos x \right) + \left(\int_0^{2\pi} f(x) \frac{1}{\sqrt{\pi}} \sin x \, dx \right) \left(\frac{1}{\sqrt{\pi}} \sin x \right)$$

and then rearranging gives us

$$f = \underbrace{\left(\frac{1}{\pi} \int_0^{2\pi} f(x) \cos x \, dx \right)}_{a} (\cos x) + \underbrace{\left(\frac{1}{\pi} \int_0^{2\pi} f(x) \sin x \, dx \right)}_{b} (\sin x)$$

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5. *(15 pts)* Use the arithmetic given below to find a fundamental set of solutions to the system $\vec{y}' = A\vec{y}$.

$$A = \begin{pmatrix} 5 & -3 \\ -13 & 8 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 8 & 3 \\ 13 & 5 \end{pmatrix}$$

$$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\beta}^{\mathcal{S}} [T]_{\beta}^{\beta} [I]_{\mathcal{S}}^{\beta}$$

So the eigenvectors are $\begin{pmatrix} 5 \\ -13 \end{pmatrix}, \begin{pmatrix} -3 \\ 8 \end{pmatrix}$, and the eigenvalues are $4, 3$, respectively.

The theorem from class then gives us the f.s.s.

$$\left\{ e^{4x}\begin{pmatrix} 5 \\ -13 \end{pmatrix}, e^{3x}\begin{pmatrix} -3 \\ 8 \end{pmatrix} \right\}$$

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6. *(20 pts)* Use the arithmetic given below to find a fundamental set of solutions to the system $\vec{y}' = B\vec{y}$.

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -13 & 8 \end{pmatrix} \begin{pmatrix} & B & \end{pmatrix} \begin{pmatrix} 8 & 3 \\ 13 & 5 \end{pmatrix}$$

$$[T]_{\gamma}^{\gamma} = [I]_{\delta}^{\gamma} [T]_{\delta}^{\delta} [I]_{\gamma}^{\delta}$$

So a Jordan basis is $\left\{ \begin{pmatrix} 8 \\ 13 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\}$, and there is exactly one basic Jordan block in the Jordan form.

$$e^{xA}\vec{v}_1 = e^{2x}\vec{v}_1 \qquad = e^{2x}\begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

$$e^{xA}\vec{v}_2 = e^{2x}(\vec{v}_2 + x\vec{v}_1) \quad = e^{2x}\begin{pmatrix} 3 + 8x \\ 5 + 13x \end{pmatrix}$$

This pair is a f.s.s. .

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