EXAM 2

Math 216, 2018 Fall, Clark Bray.

Name: Solutions _______________ Section: ______ Student ID: ____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators. Scratch paper is allowed, but (1) it must be from the instructor, (2) it must be returned with the exam, and (3) it will NOT be graded.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything on the QR codes or near the staple.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can ONLY be done on the front or back of the page the question is written on.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

(Nothing on this page will be graded!)

1. *(20 pts)* The reduced row echelon form of the matrix A is given below.

$$\begin{pmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find a basis for the null space of A.

$$1x_1 + 2x_2 + 0x_3 + 5x_4 = 0$$
$$0x_1 + 0x_2 + 1x_3 + 6x_4 = 0$$
$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 - 5x_4 \\ x_2 \\ -6x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

So $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -6 \\ 1 \end{pmatrix} \right\}$ is a basis for $NS(A)$.

(b) Find a basis for the row space of A.

The pivot rows of rref(A) are such a basis:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 6 \end{pmatrix} \right\}$$

(c) The matrix A has columns $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ (in that order). Find a significant relation among these columns of A and explain how you arrived at your answer.

Column relations are preserved by row operations, so as is evident from the above columns of rref(A) we have

$$5\vec{a}_1 + 0\vec{a}_2 + 6\vec{a}_3 - 1\vec{a}_4 = \vec{0}$$

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2. *(15 pts)* Bob is interested in deciding if the list of functions $\{\cos x + \sin x, \sin x + x\cos x, x\cos x + x\sin x, x\sin x + \cos x\}$ is linearly independent or linearly dependent. He has begun a computation of the Wronskian, and has correctly derived it to:

$$\det\begin{pmatrix} \cos x + \sin x & \sin x + x\cos x & x\cos x + x\sin x & x\sin x + \cos x \\ -\sin x + \cos x & 2\cos x - x\sin x & \cos x - x\sin x + \sin x + x\cos x & x\cos x \\ -\cos x - \sin x & -3\sin x - x\cos x & -2\sin x - x\cos x + 2\cos x - x\sin x & \cos x - x\sin x \\ \sin x - \cos x & -4\cos x + x\sin x & -3\cos x + x\sin x - 3\sin x - x\cos x & -2\sin x - x\cos x \end{pmatrix}$$

Understandably, he very much does not want to work out this determinant. Can you help him to use the Wronskian to decide if this list is linearly independent? (Be sure that your answer makes significant use of the Wronskian. Explain all of your reasoning.)

These functions are all solutions to $L(y) = 0$ with

$$p(\lambda) = (\lambda - i)^2(\lambda + i)^2 = \lambda^4 + 2\lambda^2 + 1$$

so

$$y'''' + 2y'' + y = 0$$

This is of order 4, and satisfies all conditions of the existence & uniqueness theorem, so for these functions the Wronskian need only be checked at a single value of x.

$$w(0) = \det\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ -1 & 0 & 2 & 1 \\ -1 & -4 & -3 & 0 \end{pmatrix} = 1\det\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -4 & -3 & 0 \end{pmatrix} - 1\det\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ -1 & -4 & -3 \end{pmatrix}$$

$$= 0$$

So this list of functions is <u>linearly dependent.</u>

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3. *(15 pts)* Find a fundamental set of real solutions to the linear differential equation $L(y) = 0$ whose partially factored characteristic polynomial is given below.

$$p(\lambda) = (\lambda + 2)^2(\lambda^3 - 3\lambda + 2)(\lambda^2 + 6\lambda + 25)^2$$

$\lambda^3 - 3\lambda + 2$ has poss. rat'l roots $\pm 1, \pm 2$. -2 is a root, so $(\lambda+2)$ is a factor.

$$\begin{array}{r|l} & \lambda^2 - 2\lambda + 1 \\ \hline \lambda+2 & \lambda^3 + 0\lambda^2 - 3\lambda + 2 \\ & \lambda^3 + 2\lambda^2 \\ \hline & -2\lambda^2 - 3\lambda + 2 \\ & -2\lambda^2 - 4\lambda \\ \hline & \lambda + 2 \\ & \lambda + 2 \\ \hline & 0 \end{array}$$

$\Rightarrow \lambda^3 - 3\lambda + 2 = (\lambda+2)(\lambda-1)^2$

$\lambda^2 + 6\lambda + 25 = 0$

$(\lambda+3)^2 + 16 = 0$

$\Rightarrow \lambda = -3 \pm 4i$

So $p(\lambda) = (\lambda+2)^3(\lambda-1)^2(\lambda-(-3+4i))^2(\lambda-(-3-4i))^2$

By the theorem from class, a real fundamental set of solutions is

$$\{e^{-2x}, xe^{-2x}, x^2e^{-2x}, e^x, xe^x, e^{-3x}\cos 4x, e^{-3x}\sin 4x, xe^{-3x}\cos 4x, xe^{-3x}\sin 4x\}$$

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4. *(15 pts)* Find the form of a particular solution to the equation $L(y) = x^2e^x - xe^{-x}\cos(2x)$, whose partially factored characteristic polynomial is given below.

$$p(\lambda) = (\lambda^2 + 2\lambda + 5)^3$$

$\lambda^2+2\lambda+5=0 \Rightarrow (\lambda+1)^2+4=0 \Rightarrow r=-1\pm 2i$ are roots; in $p(\lambda)$ multiplicity $=3$

For $L(y)=g_1=x^2e^x$, we have $r=1$ is <u>not</u> a root of p.

So $y_1=(Ax^2+Bx+C)e^x$

For $L(y)=g_2=-xe^{-x}\cos 2x$, we have $r=-1+2i$, <u>is</u> a root of p, with $m=3$.

So
$$y_2 = x^3(d_1x+d_2)e^{-x}\cos 2x + x^3(f_1x+f_2)e^{-x}\sin 2x$$

Then the particular solution has the form

$$y_p = (Ax^2+Bx+C)e^x + x^3(d_1x+d_2)e^{-x}\cos 2x + x^3(f_1x+f_2)e^{-x}\sin 2x$$

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5. *(15 pts)* Find the gain and phase shift in the physical system represented by the differential equation below.

$$y'' + y' + 3y = 2\cos(2t)$$

We consider the associated complex equation

$$z'' + z' + 3z = 2e^{2it}$$

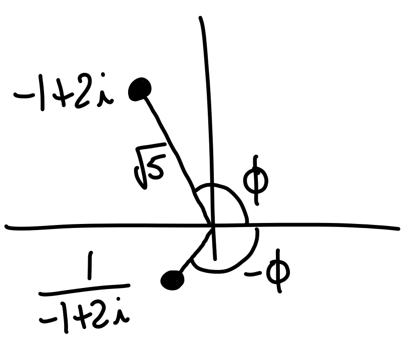
$r = 2i$ is not a root of $p(\lambda) = \lambda^2 + \lambda + 3$, so we guess

$$z = T \cdot 2e^{2it}$$

and the equation becomes

$$T \cdot \left(-8e^{2it}\right) + T \cdot \left(4ie^{2it}\right) + 3T \cdot 2e^{2it} = 2e^{2it}$$

$$\Rightarrow -8T + 4iT + 6T = 2, \quad T = \frac{1}{-1+2i} = Ge^{-i\phi}$$



$$G = \frac{1}{\|1+2i\|} = \frac{1}{\sqrt{5}}$$

$$\phi = \arccos\left(\frac{-1}{\sqrt{5}}\right)$$

So $z = \left(Ge^{-i\phi}\right) 2e^{2it} = 2Ge^{i(2t-\phi)}$

and $y = \mathrm{Re}(z) = 2G\cos(2t-\phi)$

Gain $= \frac{2G}{2} = G = \frac{1}{\sqrt{5}}$, phase shift $= \phi = \arccos\left(\frac{-1}{\sqrt{5}}\right)$

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6. *(20 pts)* The linear transformation $T : P_4 \to C^0$ has

$$T(1) = \sin x, \quad T(x) = e^x, \quad T(x^2) = 4\sin x - e^x, \quad T(x^3) = e^x - \sin x, \quad T(x^4) = 3e^x + 2\sin x$$

(a) Find $T\Big((x+1)^3\Big)$. $= T\left(x^3+3x^2+3x+1\right)$

$= T(x^3) + 3T(x^2) + 3T(x) + T(1)$

$= \left(e^x - \sin x\right) + 3\left(4\sin x - e^x\right) + 3\left(e^x\right) + 1\left(\sin x\right)$

$= e^x + 12\sin x$

(b) Find $\dim(\mathrm{im}(T))$.

$\{1, x, x^2, x^3, x^4\}$ is a basis for P_4, so $\mathrm{im}(T)$ is the span of those images, all of which are in $\mathrm{span}(e^x, \sin x)$ which is two dimensional.

The given images are not all zero nor all multiples of each other, so $\dim(\mathrm{im}(T)) = 2$.

(c) Find $\dim(\ker(T))$.

$\dim(\ker(T)) + \dim(\mathrm{im}(T)) = \dim(P_4)$

$\dim(\ker(T)) + 2 = 5$

So $\dim(\ker(T)) = 3$

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