EXAM 1

Math 216, 2018 Fall, Clark Bray.

Name: Solutions ______ Section:______ Student ID:______

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators. Scratch paper is allowed, but (1) it must be from the instructor, (2) it must be returned with the exam, and (3) it will NOT be graded.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything on the QR codes or near the staple.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can ONLY be done on the front or back of the page the question is written on.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______

(Nothing on this page will be graded!)

1. *(20 pts)* The matrix A is row reduced to its reduced row echelon form R by the nonsingular matrix E, both given below.

$$E = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find the complete set of solutions to the system $A\vec{x} = \underbrace{(3, 0, -1)}_{\vec{b}}$.

We multiply both sides of $A\vec{x} = \vec{b}$ by E to row reduce to

$$R\vec{x} = E\vec{b}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 9 \\ 0 & 0 & 1 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \to x_1 = 9 - 2x_2 - 4x_4 \\ \to x_3 = 5 - 5x_4 \end{array}$$

Then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 - 2x_2 - 4x_4 \\ x_2 \\ 5 - 5x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ 5 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ -5 \\ 1 \end{pmatrix}$

(b) Find condition(s) on the coordinates of $\vec{b} = (b_1, b_2, b_3)$ for the equation $A\vec{x} = \vec{b}$ to have at least one solution.

We need the third coordinate of $E\vec{b}$ to be zero, so

$$\boxed{1b_1 + 2b_2 + 3b_3 = 0}$$

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2. *(15 pts)*

(a) Find the matrix L with the property that, for any 3×3 matrix M with rows M_1, M_2, M_3, the rows of LM are always $M_3 - 3M_1, M_1 + 2M_2, 5M_2 + 4M_3$.

$$\underbrace{\begin{pmatrix} -3 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 5 & 4 \end{pmatrix}}_{L} \begin{pmatrix} \text{---} M_1 \text{---} \\ \text{---} M_2 \text{---} \\ \text{---} M_3 \text{---} \end{pmatrix} = \begin{pmatrix} -3M_1 + 0M_2 + 1M_3 \\ 1M_1 + 2M_2 + 0M_3 \\ 0M_1 + 5M_2 + 4M_3 \end{pmatrix}$$

(b) Find the matrix R with the property that, for any 3×3 matrix M with columns m_1, m_2, m_3, the columns of MR are always $4m_1 + 5m_2 + 6m_3, 7m_1 + 8m_2 + 9m_3, m_1 - m_2 - m_3$.

$$\begin{pmatrix} | & | & | \\ m_1 & m_2 & m_3 \\ | & | & | \end{pmatrix} \underbrace{\begin{pmatrix} 4 & 7 & 1 \\ 5 & 8 & -1 \\ 6 & 9 & -1 \end{pmatrix}}_{R} = \begin{pmatrix} 4m_1 & 7m_1 & 1m_1 \\ + & + & + \\ 5m_2 & 8m_2 & -1m_2 \\ + & + & + \\ 6m_3 & 9m_3 & -1m_3 \end{pmatrix}$$

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3. *(15 pts)*

(a) Compute the determinant below with as little arithmetic as possible.

$$\det\begin{pmatrix} 123 & 1 & 123 & 0 \\ 234 & 0 & 234 & 1 \\ 345 & 1 & 345 & 1 \\ 456 & 789 & 457 & 678 \end{pmatrix}$$

transpose

$$= \det\begin{pmatrix} 123 & 234 & 345 & 456 \\ 1 & 0 & 1 & 789 \\ 123 & 234 & 345 & 457 \\ 0 & 1 & 1 & 678 \end{pmatrix}$$

$$= \det\begin{pmatrix} 123 & 234 & 345 & 456 \\ 1 & 0 & 1 & 789 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 678 \end{pmatrix} \quad ③-① \leftarrow \text{preserves det!}$$

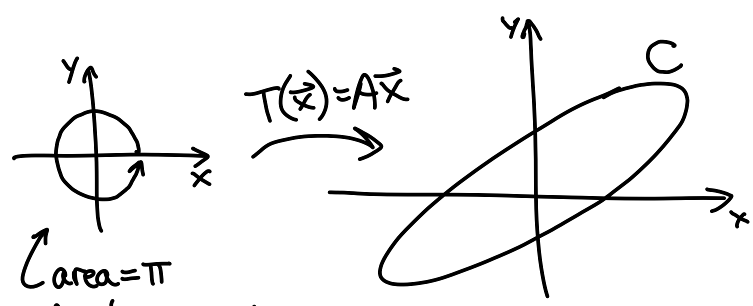
$$= -\det\begin{pmatrix} 123 & 234 & 345 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \leftarrow \text{by cofactor expansion along 3rd row}$$

$$= -\left(-\det\begin{pmatrix} 234 & 345 \\ 1 & 1 \end{pmatrix} - \det\begin{pmatrix} 123 & 234 \\ 0 & 1 \end{pmatrix}\right)$$

$$= -111 + 123 = \boxed{12}$$

(b) The curve C parametrized by $\vec{x}(t) = (3\sin t - 4\cos t, 5\sin t - 11\cos t)$ can be viewed as the image of the more familiar curve $(\cos t, \sin t)$ (which goes counterclockwise around the unit circle) by way of a convenient function. Use this idea to compute the area enclosed by C and decide if the parametrization goes around that area clockwise or counterclockwise.

$$\vec{X}(t) = \begin{pmatrix} 3\sin t - 4\cos t \\ 5\sin t - 11\cos t \end{pmatrix} = \underbrace{\begin{pmatrix} -4 & 3 \\ -11 & 5 \end{pmatrix}}_{A}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$



area $= \pi$
direction: ccwise

$\det A = 13$, so $T(\vec{x}) = A\vec{x}$ stretches areas by 13 and does <u>not</u> flip.

So C encloses an area of $\boxed{13\pi}$ and goes around ccwise.

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4. *(15 pts)* We know that

$$\det\begin{pmatrix}\vec{v}\\ \vec{a}\\ \vec{w}\end{pmatrix} = 1$$

(a) Use multilinearity to compute $\det\begin{pmatrix}3\vec{v}-5\vec{w}\\ \vec{a}\\ \vec{w}\end{pmatrix}$

$$= 3\det\begin{pmatrix}\vec{v}\\ \vec{a}\\ \vec{w}\end{pmatrix} - 5\det\begin{pmatrix}\vec{w}\\ \vec{a}\\ \vec{w}\end{pmatrix}$$

$$= 3\cdot 1 - 5\cdot 0$$

$$= 3$$

(b) Use multilinearity to compute $\det\begin{pmatrix}4\vec{v}-3\vec{w}\\ \vec{a}\\ 5\vec{v}-7\vec{w}\end{pmatrix}$

$$= 4\det\begin{pmatrix}\vec{v}\\ \vec{a}\\ 5\vec{v}-7\vec{w}\end{pmatrix} - 3\det\begin{pmatrix}\vec{w}\\ \vec{a}\\ 5\vec{v}-7\vec{w}\end{pmatrix}$$

$$= 4\left(5\det\begin{pmatrix}\vec{v}\\ \vec{a}\\ \vec{v}\end{pmatrix} - 7\det\begin{pmatrix}\vec{v}\\ \vec{a}\\ \vec{w}\end{pmatrix}\right) - 3\left(5\det\begin{pmatrix}\vec{w}\\ \vec{a}\\ \vec{v}\end{pmatrix} - 7\det\begin{pmatrix}\vec{w}\\ \vec{a}\\ \vec{w}\end{pmatrix}\right)$$

$= 0$ by antisymmetry (first and last determinants)

$$= -28\det\begin{pmatrix}\vec{v}\\ \vec{a}\\ \vec{w}\end{pmatrix} - 15\underbrace{\det\begin{pmatrix}\vec{w}\\ \vec{a}\\ \vec{v}\end{pmatrix}}_{=-1 \text{ by antisymmetry}}$$

$$= -13$$

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5. *(15 pts)* Show that $V = \{f \in C^2[0,3] | f''(1) = 0\}$ is a vector space.

We will show that V is closed under addition and scalar multiplication in $C^2[0,3]$.

Addition:

Suppose $g, h \in V$.

Then $g''(1) = 0$, $h''(1) = 0$.

So $(g+h)''(1) = g''(1) + h''(1) = 0 + 0 = 0$

This means $g+h \in V$.

Scalar multiplication:

Suppose $g \in V$.

Then $g''(1) = 0$.

So $(cg)''(1) = c(g''(1)) = c \cdot 0 = 0$.

This means $cg \in V$.

Therefore V is a subspace of $C^2[0,3]$, and thus a vector space.

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6. *(20 pts)* Suppose that $\beta = \{\vec{a}, \vec{b}, \vec{d}\}$ is a basis for a vector space V. Use coordinates to show that the list $\{3\vec{a} - 2\vec{b}, 2\vec{a} + 4\vec{d}, \vec{b} - \vec{d}\}$ must be linearly independent.

We consider relations

$$c_1(3\vec{a}-2\vec{b}) + c_2(2\vec{a}+4\vec{d}) + c_3(\vec{b}-\vec{d}) = \vec{0}$$

Taking coordinates of both sides w.r.t. β, we get

$$c_1\begin{pmatrix}3\\-2\\0\end{pmatrix} + c_2\begin{pmatrix}2\\0\\4\end{pmatrix} + c_3\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

which we can write as a matrix equation

$$\begin{pmatrix}3 & 2 & 0\\-2 & 0 & 1\\0 & 4 & -1\end{pmatrix}\vec{c} = \vec{0}$$

The determinant of this matrix is $-16 \neq 0$, so the matrix is nonsingular and thus the trivial solution is the only solution.

So the given list of vectors has no significant relations and thus is independent.

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