EXAM 3

Math 216, 2017-2018 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number______ TA______ Day/Time______

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______

Total Score ______ (/100 points)

1. *(16 pts)* We are given that $S : V \to V$ is a linear transformation, and that the bases $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\mathcal{W} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ for V are related by

$$\begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Write $\vec{v}_3$ as a linear combination of the $\mathcal{W}$ vectors.

$$\vec{v}_3 = 4\vec{w}_1 + 3\vec{w}_2 + 1\vec{w}_3$$

(b) Find the change of basis matrices $[I]^{\mathcal{V}}_{\mathcal{W}}$ and $[I]^{\mathcal{W}}_{\mathcal{V}}$.

$$[I]^{\mathcal{W}}_{\mathcal{V}} = \begin{pmatrix} | & | & | \\ [\vec{v}_1]_{\mathcal{W}} & [\vec{v}_2]_{\mathcal{W}} & [\vec{v}_3]_{\mathcal{W}} \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} ① \\ ② - 3③ \\ ③ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} ① - 2② - 4③ \\ ② \\ ③ \end{array}$$

$$\underbrace{\begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}}_{= \left([I]^{\mathcal{W}}_{\mathcal{V}}\right)^{-1} = [I]^{\mathcal{V}}_{\mathcal{W}}}$$

(c) Suppose we know that $[\vec{x}]_{\mathcal{V}} = (2, 1, 3)$, and

$$[S]^{\mathcal{W}}_{\mathcal{W}} = \begin{pmatrix} 1 & -2 & 0 \\ 4 & 3 & 2 \\ 2 & -1 & 1 \end{pmatrix}$$

Find $[S(\vec{x})]_{\mathcal{V}}$.

$$\begin{tikzcd}
{[x]_{\mathcal{V}} \in \mathbb{R}^2} \arrow[r, "{[S]^{\mathcal{V}}_{\mathcal{V}}}"] \arrow[d, "{[I]^{\mathcal{W}}_{\mathcal{V}}}"'] & {\mathbb{R}^2 \ni [S(\vec{x})]_{\mathcal{V}}} \\
\mathbb{R}^2 \arrow[r, "{[S]^{\mathcal{W}}_{\mathcal{W}}}"] & \mathbb{R}^2 \arrow[u, "{[I]^{\mathcal{V}}_{\mathcal{W}}}"']
\end{tikzcd}$$

$$= [S]^{\mathcal{V}}_{\mathcal{V}} [\vec{x}]_{\mathcal{V}}$$

$$= \left([I]^{\mathcal{V}}_{\mathcal{W}} [S]^{\mathcal{W}}_{\mathcal{W}} [I]^{\mathcal{W}}_{\mathcal{V}}\right) [\vec{x}]_{\mathcal{V}}$$

$$= \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 4 & 3 & 2 \\ 2 & -1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}}_{\begin{pmatrix} 16 \\ 10 \\ 3 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 100 \\ 25 \end{pmatrix}$$

$$= \begin{pmatrix} -154 \\ 25 \\ 25 \end{pmatrix}$$

2. *(18 pts)* The matrix $A = [T]_{\mathcal{S}}^{\mathcal{S}}$ is put into the Jordan form below by the basis $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$.

$$J = [T]_{\mathcal{V}}^{\mathcal{V}} = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(a) With respect to which of the following matrices would T also be represented in a valid Jordan form? (You do not have to justify these answers.) ✓

i. $\mathcal{W}_1 = \{\vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_1\}$ Yes : $J_1 = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ ✓

ii. $\mathcal{W}_2 = \{\vec{v}_4, \vec{v}_3, \vec{v}_2, \vec{v}_1\}$ No : $\longrightarrow J_2 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ ✗

iii. $\mathcal{W}_3 = \{\vec{v}_1, \vec{v}_4, \vec{v}_2, \vec{v}_3\}$ Yes : $\longrightarrow J_3 = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ ✓

(b) Write out four equations telling you how the images of the $\mathcal{V}$ vectors are combinations of the $\mathcal{V}$ vectors.

$$T(\vec{v}_1) = 5\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4$$
$$T(\vec{v}_2) = 0\vec{v}_1 + 3\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4$$
$$T(\vec{v}_3) = 0\vec{v}_1 + 1\vec{v}_2 + 3\vec{v}_3 + 0\vec{v}_4$$
$$T(\vec{v}_4) = 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + 3\vec{v}_4$$

(c) Rewrite the above equations naturally with respect to the basis $\mathcal{W} = \{\vec{v}_4, \vec{v}_2, \vec{v}_3, \vec{v}_1\}$ and deduce the matrix $J' = [T]_{\mathcal{W}}^{\mathcal{W}}$.

$$T(\vec{v}_4) = 3\vec{v}_4 + 0\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_1$$
$$T(\vec{v}_2) = 0\vec{v}_4 + 3\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_1$$
$$T(\vec{v}_3) = 0\vec{v}_4 + 1\vec{v}_2 + 3\vec{v}_3 + 0\vec{v}_1$$
$$T(\vec{v}_1) = 0\vec{v}_4 + 0\vec{v}_2 + 0\vec{v}_3 + 5\vec{v}_1$$

$$\Rightarrow [T]_{\mathcal{W}}^{\mathcal{W}} = \begin{pmatrix} [T(\vec{v})]_{\mathcal{W}} & [T(\vec{v})]_{\mathcal{W}} & [T(\vec{v})]_{\mathcal{W}} & [T(\vec{v})]_{\mathcal{W}} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

3. *(18 pts)* Perform Gram-Schmidt orthonormalization on the pair of vectors $\{1, x\}$ in the inner product space $C([0,2])$ with the inner product

$\{1, x\}$: $1 = w_1$, $x = w_2$

$$\langle f, g\rangle = \int_0^2 6x^2\, f(x)\, g(x)\, dx$$

$$\langle w_1, w_1\rangle = \int_0^2 6x^2 \cdot 1 \cdot 1\, dx = \left(2x^3\right]_0^2 = 16$$

$$\|w_1\| = \sqrt{\langle w_1, w_1\rangle} = 4, \quad \text{so} \quad v_1 = \frac{w_1}{\|w_1\|} = \frac{1}{4}$$

$$\langle w_2, v_1\rangle = \int_0^2 6x^2 \cdot x \cdot \tfrac{1}{4}\, dx = \int_0^2 \tfrac{3}{2}x^3\, dx = \left(\tfrac{3}{8}x^4\right]_0^2 = 6$$

$$x_2 = w_2 - \langle w_2, v_1\rangle v_1 = x - (6)\left(\tfrac{1}{4}\right) = x - \tfrac{3}{2}$$

$$\langle x_2, x_2\rangle = \int_0^2 6x^2 \cdot \left(x - \tfrac{3}{2}\right)\left(x - \tfrac{3}{2}\right) dx$$

$$= \int_0^2 6x^4 - 18x^3 + \tfrac{27}{2}x^2\, dx$$

$$= \tfrac{6}{5}x^5 - \tfrac{9}{2}x^4 + \tfrac{9}{2}x^3\Big]_0^2 = \tfrac{192}{5} - 72 + 36$$

$$= \tfrac{12}{5}$$

$$\|x_2\| = \sqrt{\langle x_2, x_2\rangle} = \sqrt{12/5}, \quad \text{so} \quad v_2 = \frac{x_2}{\|x_2\|} = \frac{x - 3/2}{\sqrt{12/5}}$$

$$= x\sqrt{5/12} - \frac{\sqrt{15}}{4}$$

So the orthonormalized basis is $\left\{ \frac{1}{4},\ x\sqrt{5/12} - \frac{\sqrt{15}}{4} \right\}$.

4. *(16 pts)* Find the inverse of the matrix A below by using the fact that there is a diagonal matrix D for which AD is easy to invert. *(Hint: What is special about the columns of A?)*

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & -2 & -1 \\ 0 & 1 & -10 \end{pmatrix}$$

Columns of A are orthogonal. We can make an orthogonal matrix AD then by

$$D = \begin{pmatrix} 1/\sqrt{5} & 0 & 0 \\ 0 & 1/\sqrt{21} & 0 \\ 0 & 0 & 1/\sqrt{105} \end{pmatrix} \Rightarrow AD = \begin{pmatrix} 1/\sqrt{5} & 4/\sqrt{21} & 2/\sqrt{105} \\ 2/\sqrt{5} & -2/\sqrt{21} & -1/\sqrt{105} \\ 0 & 1/\sqrt{21} & 10/\sqrt{105} \end{pmatrix}$$

Being orthogonal, we can easily invert AD since $(AD)^{-1} = (AD)^T$

$$(AD)^{-1} = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 4/\sqrt{21} & -2/\sqrt{21} & 1/\sqrt{21} \\ 2/\sqrt{105} & -1/\sqrt{105} & 10/\sqrt{105} \end{pmatrix} = D^{-1}A^{-1}$$

Then $A^{-1} = D\,(AD)^{-1}$

$$= \begin{pmatrix} 1/\sqrt{5} & 0 & 0 \\ 0 & 1/\sqrt{21} & 0 \\ 0 & 0 & 1/\sqrt{105} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 4/\sqrt{21} & -2/\sqrt{21} & 1/\sqrt{21} \\ 2/\sqrt{105} & -1/\sqrt{105} & 10/\sqrt{105} \end{pmatrix}$$

$$= \begin{pmatrix} 1/5 & 2/5 & 0 \\ 4/21 & -2/21 & 1/21 \\ 2/105 & -1/105 & 10/105 \end{pmatrix}$$

5. *(16 pts)* Find a fundamental set of solutions to the system of differential equations described by $\vec{y}' = A\vec{y}$, where

$$A = \begin{pmatrix} 3 & -7 & -3 \\ -26 & -86 & -39 \\ 62 & 217 & 98 \end{pmatrix}$$

Here is some arithmetic that you might find useful:

$$\begin{pmatrix} 3 & -7 & -3 \\ -26 & -86 & -39 \\ 62 & 217 & 98 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ -3 & 1 & -13 \\ 7 & -2 & 31 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 5 & 2 & 1 \\ 2 & 7 & 3 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ -3 & 1 & -13 \\ 7 & -2 & 31 \end{pmatrix} \begin{pmatrix} 5 & 2 & 1 \\ 2 & 7 & 3 \\ -1 & 0 & 0 \end{pmatrix}$$

This shows that this is a conjugation, so this is the Jordan form

Interpreting the conjugation as

$$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{V}}^{\mathcal{S}} [T]_{\mathcal{V}}^{\mathcal{V}} [I]_{\mathcal{S}}^{\mathcal{V}}$$

We see that the Jordan basis vectors are the columns of this matrix.

$$\mathcal{V} = \left\{ \underset{\vec{v}_1}{\underset{\parallel}{\begin{pmatrix} 0 \\ -3 \\ 7 \end{pmatrix}}}, \underset{\vec{v}_2}{\underset{\parallel}{\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}}}, \underset{\vec{v}_3}{\underset{\parallel}{\begin{pmatrix} -1 \\ -13 \\ 31 \end{pmatrix}}} \right\}$$

A fundamental set of solutions is thus

$$\left\{ e^{xA}\vec{v}_1,\ e^{xA}\vec{v}_2,\ e^{xA}\vec{v}_3 \right\}$$

which are evaluated as

$$e^{xA}\vec{v}_1 = e^{5x}\vec{v}_1 = e^{5x}\begin{pmatrix} 0 \\ -3 \\ 7 \end{pmatrix}$$

$$e^{xA}\vec{v}_2 = e^{5x}\vec{v}_2 = e^{5x}\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$e^{xA}\vec{v}_3 = e^{5x}\left(\vec{v}_3 + x\vec{v}_2\right) = e^{5x}\begin{pmatrix} -1 \\ -13 + 1x \\ 31 - 2x \end{pmatrix}$$

6. *(16 pts)*

(a) Find a particular solution to the first order system below.

$$\begin{aligned} y_1' &= 2y_1 + y_2 + e^{2ix} \\ y_2' &= y_1 + 2y_2 \end{aligned}$$

$$\vec{Y}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \vec{Y} + \begin{pmatrix} e^{2ix} \\ 0 \end{pmatrix}$$

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is symmetric so $2i$ is not an eigenvalue. So we guess

$$\vec{Y} = e^{2ix} \vec{a}$$

and the equation becomes

$$2i e^{2ix} \vec{a} = A e^{2ix} a + e^{2ix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow (A - 2iI)\vec{a} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-2i & 1 \\ 1 & 2-2i \end{pmatrix} \vec{a} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow \vec{a} = \frac{\begin{pmatrix} 2-2i & -1 \\ -1 & 2-2i \end{pmatrix}}{-1-8i} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2-2i \\ -1 \end{pmatrix} / 1+8i \cdot \frac{1-8i}{1-8i} = \begin{pmatrix} -14-18i \\ -1+8i \end{pmatrix} / 65$$

So $\vec{Y}_p = e^{2ix} \begin{pmatrix} -14-18i \\ -1+8i \end{pmatrix} / 65$

(b) Find a particular solution to the first order system below.

$$\begin{aligned} u_1' &= 2u_1 + u_2 + \cos(2x) \\ u_2' &= u_1 + 2u_2 \end{aligned}$$

The above equation is the associated complex equation, so

$$\vec{u}_p = \mathrm{Re}(\vec{Y}_p) = \mathrm{Re}\left(e^{2ix} \begin{pmatrix} -14-18i \\ -1+8i \end{pmatrix} / 65 \right)$$

$$= \mathrm{Re}\left((\cos 2x + i \sin 2x) \begin{pmatrix} -14-18i \\ -1+8i \end{pmatrix} / 65 \right)$$

$$= \begin{pmatrix} -14\cos 2x + 18 \sin 2x \\ -1 \cos 2x - 8 \sin 2x \end{pmatrix} / 65$$