EXAM 3

Math 216, 2017-2018 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name _____

Disc.:   Number_____   TA_____   Day/Time_____

"I have adhered to the Duke Community
Standard in completing this
examination."

1. _____

2. _____

Signature: _____

3. _____

4. _____

5. _____

6. _____

Total Score _____ (/100 points)

1. *(16 pts)* We are given that $S : V \to V$ is a linear transformation, and that the bases $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $W = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ for V are related by

$$\begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Write $\vec{v}_3$ as a linear combination of the W vectors.

(b) Find the change of basis matrices $[I]_W^V$ and $[I]_V^W$.

(c) Suppose we know that $[\vec{x}]_V = (2, 1, 3)$, and

$$[S]_W^W = \begin{pmatrix} 1 & -2 & 0 \\ 4 & 3 & 2 \\ 2 & -1 & 1 \end{pmatrix}$$

Find $[S(\vec{x})]_V$.

2. *(18 pts)* The matrix $A = [T]_{\mathcal{S}}^{\mathcal{S}}$ is put into the Jordan form below by the basis $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$.

$$J = [T]_{\mathcal{V}}^{\mathcal{V}} = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(a) With respect to which of the following matrices would T also be represented in a valid Jordan form? (You do not have to justify these answers.)

   i. $\mathcal{W}_1 = \{\vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_1\}$

   ii. $\mathcal{W}_2 = \{\vec{v}_4, \vec{v}_3, \vec{v}_2, \vec{v}_1\}$

   iii. $\mathcal{W}_3 = \{\vec{v}_1, \vec{v}_4, \vec{v}_2, \vec{v}_3\}$

(b) Write out four equations telling you how the images of the $\mathcal{V}$ vectors are combinations of the $\mathcal{V}$ vectors.

(c) Rewrite the above equations naturally with respect to the basis $\mathcal{W} = \{\vec{v}_4, \vec{v}_2, \vec{v}_3, \vec{v}_1\}$ and deduce the matrix $J' = [T]_{\mathcal{W}}^{\mathcal{W}}$.

3. *(18 pts)* Perform Gram-Schmidt orthonormalization on the pair of vectors $\{1, x\}$ in the inner product space $C([0,2])$ with the inner product

$$\langle f, g \rangle = \int_0^2 6x^2 \, f(x) \, g(x) \, dx$$

$$\langle f, g \rangle = \frac{3}{}$$

4. *(16 pts)* Find the inverse of the matrix A below by using the fact that there is a diagonal matrix D for which AD is easy to invert. *(Hint: What is special about the columns of A?)*

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & -2 & -1 \\ 0 & 1 & -10 \end{pmatrix}$$

$$A = \quad \begin{matrix} 4 & -2 & -1 \\ 0 & 1 & -10 \end{matrix}$$

5. *(16 pts)* Find a fundamental set of solutions to the system of differential equations described by $\vec{y}' = A\vec{y}$, where

$$A = \begin{pmatrix} 3 & -7 & -3 \\ -26 & -86 & -39 \\ 62 & 217 & 98 \end{pmatrix}$$

Here is some arithmetic that you might find useful:

$$\begin{pmatrix} 3 & -7 & -3 \\ -26 & -86 & -39 \\ 62 & 217 & 98 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ -3 & 1 & -13 \\ 7 & -2 & 31 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 5 & 2 & 1 \\ 2 & 7 & 3 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ -3 & 1 & -13 \\ 7 & -2 & 31 \end{pmatrix} \begin{pmatrix} 5 & 2 & 1 \\ 2 & 7 & 3 \\ -1 & 0 & 0 \end{pmatrix}$$

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6. *(16 pts)*

(a) Find a particular solution to the first order system below.

$$\begin{aligned} y_1' &= 2y_1 + y_2 + e^{2ix} \\ y_2' &= y_1 + 2y_2 \end{aligned}$$

(b) Find a particular solution to the first order system below.

$$\begin{aligned} u_1' &= 2u_1 + u_2 + \cos(2x) \\ u_2' &= u_1 + 2u_2 \end{aligned}$$