EXAM 2

Math 216, 2017-2018 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number______ TA______ Day/Time______

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______

Total Score ______ (/100 points)

1. *(16 pts)* Show that the image of the linear transformation $T : V \to W$ is a subspace of W.

(i) We show that $\operatorname{im}(T)$ is closed under addition:

Say $w_1, w_2 \in \operatorname{im}(T)$.

$\Rightarrow w_1 = T(v_1),\ w_2 = T(v_2)$, for some $v_1, v_2 \in V$.

$$\Rightarrow w_1 + w_2 = T(v_1) + T(v_2)$$
$$= T(v_1 + v_2)$$

So $w_1 + w_2 \in \operatorname{im}(T)$

(ii) We show that $\operatorname{im}(T)$ is closed under scalar multiplication:

Say $w \in \operatorname{im}(T)$, $c \in \mathbb{R}$.

$\Rightarrow w = T(v)$ for some $v \in V$.

$$\Rightarrow cw = cT(v)$$
$$= T(cv)$$

So $cw \in \operatorname{im}(T)$.

So $\operatorname{im}(T)$ is a subspace of W.

2. *(18 pts)*

(a) Find bases for the null space, row space, and column space of the matrix

$$A = \begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} \boxed{1} & 0 & 5 & 0 \\ 0 & \boxed{1} & 7 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}}_{\text{rref}(A)} \begin{array}{l} ① - 3③ \\ ② - 4③ \\ ③ \end{array}$$

pivots

Nullspace: pivot eq's say

$$\begin{aligned} x_1 &= -5x_3 \\ x_2 &= -7x_3 \\ x_4 &= 0 \end{aligned} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5x_3 \\ -7x_3 \\ x_3 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -5 \\ -7 \\ 1 \\ 0 \end{pmatrix}$$

So $\left\{ \begin{pmatrix} -5 \\ -7 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for the null space.

Row space: Pivot rows of rref(A) gives us the basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Column space: Pivot columns of A gives us the basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \right\}$$

(b) The $x \times y$ matrix B has a 3-dimensional row space and a 5-dimensional null space. What can you conclude about x and/or y, if anything? What can you conclude about the dimension of the column space, if anything?

$\dim(RS) + \dim(NS) = \#$ of columns, so $y = 3+5 = 8$.

We do not have enough information to find x.

$\dim(CS) = \dim(RS) = 3$.

3. *(18 pts)* Find a fundamental set of real solutions to the differential equation $L(y) = 0$ whose partially factored characteristic polynomial is given below.

$$p(\lambda) = (\overbrace{\lambda^3 + 5\lambda^2 + 9\lambda + 5}^{r(x)})^3$$

Poss. rat'l roots of r : $\pm 1, \pm 5$. $r(-1) = 0 \Rightarrow \lambda + 1$ is a factor.

$$\begin{array}{r|l} & \lambda^2 + 4\lambda + 5 \\ \hline \lambda+1 & \lambda^3 + 5\lambda^2 + 9\lambda + 5 \\ & \underline{\lambda^3 + \lambda^2} \\ & \quad 4\lambda^2 + 9\lambda + 5 \\ & \quad \underline{4\lambda^2 + 4\lambda} \\ & \qquad 5\lambda + 5 \\ & \qquad \underline{5\lambda + 5} \\ & \qquad\quad 0 \end{array}$$

$$\Rightarrow r(\lambda) = (\lambda+1)(\lambda^2 + 4\lambda + 5)$$
$$= (\lambda+1)(\lambda - (-2+i))(\lambda - (-2-i))$$

roots of p : -1, $-2 \pm i$

multiplicity $= 3$

A f.s.s. then is

$$\{ e^{-x}, xe^{-x}, x^2e^{-x},$$
$$e^{-2x}\cos x, xe^{-2x}\cos x, x^2e^{-2x}\cos x,$$
$$e^{-2x}\sin x, xe^{-2x}\sin x, x^2e^{-2x}\sin x \}$$

4. *(16 pts)* Find the form of a particular solution to the differential equation below. (You do not have to evaluate the coefficients.)

$$y'' + 2y' + 10y = xe^{-x}\cos(3x) + x^2e^{4x}\sin(2x)$$

$p(\lambda) = \lambda^2 + 2\lambda + 10$, roots are $-1 \pm 3i$, multiplicities $= 1$.

For $xe^{-x}\cos 3x$, naive guess is

$$(c_1x + c_0)e^{-x}\cos 3x + (d_1x + d_0)e^{-x}\sin 3x$$

But $r = -1+3i$ <u>is</u> a root of p, with $m=1$, so the correct form is

$$(c_1x^2 + c_0x)e^{-x}\cos 3x + (d_1x^2 + d_0x)e^{-x}\sin 3x$$

For $x^2e^{4x}\sin 2x$, naive guess is

$$(f_2x^2 + f_1x + f_0)e^{4x}\cos 2x + (g_2x^2 + g_1x + g_0)e^{4x}\sin 2x$$

And $r = 4+2i$ is not a root of p so our naive guess is right.

Combined then, the form of a particular solution is

$$y_p = (c_1x^2 + c_0x)e^{-x}\cos 3x + (d_1x^2 + d_0x)e^{-x}\sin 3x$$
$$+ (f_2x^2 + f_1x + f_0)e^{4x}\cos 2x + (g_2x^2 + g_1x + g_0)e^{4x}\sin 2x$$

5. *(16 pts)* For the differential equation below representing a mass on a spring with a sinusoidal external force, a particular solution is $y_p = -\cos(3t)$.

$$y'' + 8y = \cos(3t)$$

Using this, derive

(a) the general homogeneous solution;

(b) the IVP solution with $y(0) = 0$ and $y'(0) = 0$,

(c) the beat frequency (f, not ω) of this IVP solution (as an arithmetic expression – not a decimal approximation).

a) $y'' + 8y = 0 \Rightarrow p(\lambda) = \lambda^2 + 8$, roots are $\pm 2\sqrt{2}\, i$.

So the general homogeneous solution is

$$y_H = c_1 \cos(2\sqrt{2}\, t) + c_2 \sin(2\sqrt{2}\, t) = A\cos(2\sqrt{2}\, t - \phi)$$

b) General solution and derivative are:

$$y = c_1 \cos(2\sqrt{2}\, t) + c_2 \sin(2\sqrt{2}\, t) - \cos(3t)$$
$$y' = -2\sqrt{2}\, c_1 \sin(2\sqrt{2}\, t) + 2\sqrt{2}\, c_2 \cos(2\sqrt{2}\, t) + 3\sin(3t)$$

At $t=0$, we have $\left.\begin{array}{l} 0 = c_1 - 1 \\ 0 = 2\sqrt{2}\, c_2 \end{array}\right\} \Rightarrow c_1 = 1,\ c_2 = 0$

So the I.V.P. solution is $y = \cos(2\sqrt{2}\, t) - \cos(3t)$

c) $\cos(a-b) = \cos a \cos b + \sin a \sin b$

$\underline{\cos(a+b) = \cos a \cos b - \sin a \sin b}$

$\cos(a-b) - \cos(a+b) = 2 \sin a \sin b$

$\left.\begin{array}{l} a-b = 2\sqrt{2}\, t \\ a+b = 3t \end{array}\right\} \Rightarrow a = \frac{3+2\sqrt{2}}{2} t,\ b = \frac{3-2\sqrt{2}}{2} t$

"amplitude" "oscillation"

$$\Rightarrow y = \cos(2\sqrt{2}\, t) - \cos(3t) = \left(2\sin\left(\frac{3-2\sqrt{2}}{2} t\right)\right)\sin\left(\frac{3+2\sqrt{2}}{2} t\right)$$

beat frequency $= 2 \cdot$ "amplitude" frequency $= 2\left(\frac{3-2\sqrt{2}}{2}\right)\Big/2\pi = \frac{3-2\sqrt{2}}{2\pi}$

6. *(16 pts)* Let $S = \text{span}(\cos(4x), \sin(4x))$ be the vector space of solutions to the differential equation $y'' + 16y = 0$, and let I be the vector space of initial values $I = \left\{ \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} \right\} = \left\{ \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right\} = \mathbb{R}^2$.

Show that the function $T : S \to I$ defined by $T(y) = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}$ is a linear transformation.

S and I are known already to be vector spaces.

Checking linearity:

$$T(c_1y_1 + c_2y_2) = \begin{pmatrix} (c_1y_1 + c_2y_2)(0) \\ (c_1y_1 + c_2y_2)'(0) \end{pmatrix}$$

$$= \begin{pmatrix} (c_1y_1 + c_2y_2)(0) \\ (c_1y_1' + c_2y_2')(0) \end{pmatrix}$$

$$= \begin{pmatrix} c_1y_1(0) + c_2y_2(0) \\ c_1y_1'(0) + c_2y_2'(0) \end{pmatrix}$$

$$= c_1 \begin{pmatrix} y_1(0) \\ y_1'(0) \end{pmatrix} + c_2 \begin{pmatrix} y_2(0) \\ y_2'(0) \end{pmatrix}$$

$$= c_1T(y_1) + c_2T(y_2)$$

So T is a linear transformation.