EXAM 1

Math 216, 2017-2018 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number______ TA______ Day/Time______

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______

Total Score ______ (/100 points)

1. *(16 pts)* Your friend Bob states the following:

> If the reduced row echelon form of a matrix A has a column with no pivot, then the system $A\vec{x} = \vec{b}$ must have infinitely many solutions.

(a) Find an explicit counterexample showing that Bob's statement is wrong.

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

This is already row reduced. The second column of rref(A) has no pivot and thus there is a free variable; but there is a contradiction in the second row, so there are actually no solutions at all.

(b) What additional condition about the reduced row echelon form would make Bob's statement true?

Existence is the only issue that needs to be fixed. This can be resolved by adding the condition that there must be a pivot in every row of rref(A).

2. *(16 pts)* Consider the following equation.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 7 & 4 \\ 5 & 1 & 6 \end{pmatrix} \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{pmatrix}$$

(a) Write the row vector (1 0 1) as a linear combination of the rows $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$.

Interpreting matrix multiplication in terms of linear combinations of rows, we get

$$(1 \ 0 \ 1) = 2\vec{r}_1 + 7\vec{r}_2 + 4\vec{r}_3$$

(b) Note that the third row of the product is the sum of the first two. Use this information to find a relation between the vectors $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$.

$$(1 \ 1 \ 0) = 1\vec{r}_1 + 3\vec{r}_2 + 2\vec{r}_3$$

$$(2 \ 1 \ 1) = 5\vec{r}_1 + 1\vec{r}_2 + 6\vec{r}_3$$

Combining these left sides as suggested, we get

$$\left(2\vec{r}_1 + 7\vec{r}_2 + 4\vec{r}_3\right) + \left(1\vec{r}_1 + 3\vec{r}_2 + 2\vec{r}_3\right) = \left(5\vec{r}_1 + 1\vec{r}_2 + 6\vec{r}_3\right)$$

and thus

$$-2\vec{r}_1 + 9\vec{r}_2 + 0\vec{r}_3 = \vec{0}$$

3. *(18 pts)* The 3×3 matrix M can be row reduced to the identity matrix by the following sequence of row operations:

   i. The first row is added to the second row.

   ii. 2 times the second row is added to the third row.

   iii. The second row is multiplied by 3.

(a) Compute M^{-1}.

We know that $(M \mid I)$ reduces to $(I \mid M^{-1})$. Doing the right half of this gives us

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} 1 \\ 2+1 \\ 3 \end{array} \Bigg\} (\text{i.})$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{array}{l} 1 \\ 2 \\ 3+2\cdot 2 \end{array} \Bigg\} (\text{ii.})$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 2 & 2 & 1 \end{pmatrix}}_{\text{So this is } M^{-1}.} \begin{array}{l} 1 \\ 3\cdot 2 \\ 3 \end{array}$$

(b) Compute $\det M$.

The given row reduction tells us that $3 \det M = \det I$.

So $\det M = 1/3$.

(c) Find matrices E_1, E_2, E_3 such that $M = E_1 E_2 E_3$.

The given row reduction is represented with row operation matrices as

$$F_3 F_2 F_1 M = I, \quad \text{with } F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So

$$\begin{aligned} M &= F_1^{-1} F_2^{-1} F_3^{-1} \\ &= E_1 E_2 E_3 \end{aligned} \quad \text{with } E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. *(18 pts)* The function $L : \mathbb{R}^3 \to \mathbb{R}^3$ is multiplication by a matrix M, and

$$L(1,0,0) = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \vec{v}_1, \quad L(0,1,0) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \vec{v}_2, \quad L(0,0,1) = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \vec{v}_3$$

The solid region R is known to have volume equal to 3. We denote the image of R by L with $L(R)$.

(a) Find the matrix M.

Columns of M are images of standard basis vectors, which are given.

So $$M = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 5 \\ 4 & 1 & 1 \end{pmatrix}$$

(b) Find the volume of $L(R)$.

Being a matrix multiplication, L stretches volumes by a factor of

$$\det M = 1(-4) - 2(-17) + 3(-1)$$
$$= 27$$

So $$\text{vol}(L(R)) = |\det M| \text{vol}(R) = 27 \cdot 3 = 81.$$

(c) Suppose the listing $\vec{w}_1$, $\vec{w}_2$, $\vec{w}_3$ is in left hand order. What if anything can you conclude about the ordering of the list $L(\vec{w}_1)$, $L(\vec{w}_2)$, $L(\vec{w}_3)$?

$\det M$ is positive, so L does not involve any reflection, and thus it preserves handedness. So $L(\vec{w}_1), L(\vec{w}_2), L(\vec{w}_3)$ is also in left hand order.

5. *(16 pts)* Do the vectors $(1, 2, 0)$, $(2, 0, 1)$, $(3, 3, 2)$ span $\mathbb{R}^3$? Show your argument directly from the definition of span, and show all of the steps of the argument.

We need solutions to the equation below for all $\vec{b} \in \mathbb{R}^3$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{b}$$

By coordinates, this is the system

$$\begin{aligned} 1c_1 + 2c_2 + 3c_3 &= b_1 \\ 2c_1 + 0c_2 + 3c_3 &= b_2 \\ 0c_1 + 1c_2 + 2c_3 &= b_3 \end{aligned}$$

which in matrix form is

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 0 & 3 & b_2 \\ 0 & 1 & 2 & b_3 \end{array}\right)$$

The determinant is

$$\det(A) = 1(-3) - 2(1) + 0(6) = -5$$

This is nonzero, so A is nonsingular, so there is a pivot in every row of rref(A), thus A has the existence property, and so the given vector equation does always have a solution.

So this list of vectors does span $\mathbb{R}^3$.

6. *(16 pts)* Your friend Bob is considering a candidate vector space V, whose vectors are those in $\mathbb{R}^2$, but with operations $\oplus$ and $\otimes$ defined by

$$\vec{v} \oplus \vec{w} = \vec{v} + \vec{w}$$
$$c \otimes \vec{v} = 3c\vec{v}$$

(The operations on the right sides of the equations above are the standard operations on $\mathbb{R}^2$.)

(a) One of the generic conditions required of vector spaces is

$c(u + v) = cu + cv$ for all $u, v \in V$, $c \in \mathbb{R}$

Show that this candidate space V satisfies this condition.

$$c \otimes (\vec{u} \oplus \vec{v}) = 3c(\vec{u} + \vec{v})$$
$$= 3c\vec{u} + 3c\vec{v}$$
$$= c \otimes \vec{u} + c \otimes \vec{v}$$

(b) Unfortunately for Bob, V is not actually a vector space, as it fails at least one of the required 8 conditions. Identify one of these failed conditions and show with a counterexample how V fails it.

One of the required conditions is $c(dv) = (cd)v \ \forall c, d \in \mathbb{R}, v \in V$ which for this candidate would require

$$c \otimes (d \otimes \vec{v}) = (cd) \otimes \vec{v}$$

which using $\mathbb{R}^n$ operations would require

$$3c(3d\vec{v}) = 3cd\vec{v}$$

which is false whenever $c, d \neq 0$ and $\vec{v} \neq \vec{0}$.

(Alt.: the condition $1\vec{v} = \vec{v}$ also fails.)