EXAM 1

Math 216, 2017-2018 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name ____________________________________

Disc.: Number____________ TA________________________ Day/Time________________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

Total Score ____________ (/100 points)

1. *(16 pts)* Your friend Bob states the following:

   If the reduced row echelon form of a matrix A has a column with no pivot, then the system $A\vec{x} = \vec{b}$ must have infinitely many solutions.

   (a) Find an explicit counterexample showing that Bob's statement is wrong.

   (b) What additional condition about the reduced row echelon form would make Bob's statement true?

2. *(16 pts)* Consider the following equation.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 7 & 4 \\ 5 & 1 & 6 \end{pmatrix} \begin{pmatrix} & \vec{r}_1 & \\ & \vec{r}_2 & \\ & \vec{r}_3 & \end{pmatrix}$$

(a) Write the row vector $(1\ 0\ 1)$ as a linear combination of the rows $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$.

(b) Note that the third row of the product is the sum of the first two. Use this information to find a relation between the vectors $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$.

3. *(18 pts)* The 3×3 matrix M can be row reduced to the identity matrix by the following sequence of row operations:

   i. The first row is added to the second row.

   ii. 2 times the second row is added to the third row.

   iii. The second row is multiplied by 3.

   (a) Compute M^{-1}.

   (b) Compute $\det M$.

   (c) Find matrices E_1, E_2, E_3 such that $M = E_1 E_2 E_3$.

4. *(18 pts)* The function $L : \mathbb{R}^3 \to \mathbb{R}^3$ is multiplication by a matrix M, and

$$L(1,0,0) = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \vec{v}_1, \quad L(0,1,0) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \vec{v}_2, \quad L(0,0,1) = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \vec{v}_3$$

The solid region R is known to have volume equal to 3. We denote the image of R by L with $L(R)$.

(a) Find the matrix M.

(b) Find the volume of $L(R)$.

(c) Suppose the listing $\vec{w}_1$, $\vec{w}_2$, $\vec{w}_3$ is in left hand order. What if anything can you conclude about the ordering of the list $L(\vec{w}_1)$, $L(\vec{w}_2)$, $L(\vec{w}_3)$?

5. *(16 pts)* Do the vectors $(1, 2, 0)$, $(2, 0, 1)$, $(3, 3, 2)$ span $\mathbb{R}^3$? Show your argument directly from the definition of span, and show all of the steps of the argument.

6. *(16 pts)* Your friend Bob is considering a candidate vector space V, whose vectors are those in $\mathbb{R}^2$, but with operations $\oplus$ and $\otimes$ defined by

$$\begin{aligned} \vec{v} \oplus \vec{w} &= \vec{v} + \vec{w} \\ c \otimes \vec{v} &= 3c\vec{v} \end{aligned}$$

(The operations on the right sides of the equations above are the standard operations on $\mathbb{R}^2$.)

(a) One of the generic conditions required of vector spaces is

$$c(u + v) = cu + cv \text{ for all } u,\ v \in V,\ c \in \mathbb{R}$$

Show that this candidate space V satisfies this condition.

(b) Unfortunately for Bob, V is not actually a vector space, as it fails at least one of the required 8 conditions. Identify one of these failed conditions and show with a counterexample how V fails it.