EXAM 3

Math 216, 2017-2018 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number______ TA______ Day/Time______

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______

Total Score ______ (/100 points)

1. *(20 pts)* We consider here the vector space V, the bases $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\beta = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, and linear transformation $T : V \to V$ with

$$\begin{aligned} \vec{v}_1 &= 1\vec{w}_1 + 1\vec{w}_2 + 1\vec{w}_3 \\ \vec{v}_2 &= 0\vec{w}_1 + 1\vec{w}_2 + 2\vec{w}_3 \\ \vec{v}_3 &= 0\vec{w}_1 + 0\vec{w}_2 + 1\vec{w}_3 \end{aligned} \quad \text{and} \quad \begin{aligned} T(\vec{v}_1) &= \vec{w}_1 \\ T(\vec{v}_2) &= \vec{w}_2 \\ T(\vec{v}_3) &= \vec{w}_3 \end{aligned}$$

(a) Find $[I]_\alpha^\beta$ and $[I]_\beta^\alpha$.

$$[I]_\alpha^\beta = \begin{pmatrix} | & | & | \\ [\vec{v}_1]_\beta & [\vec{v}_2]_\beta & [\vec{v}_3]_\beta \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

$$[I]_\beta^\alpha = \left([I]_\alpha^\beta\right)^{-1} :$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right) \begin{matrix} (1) \\ (2)-(1) \\ (3)-(1) \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \begin{matrix} (1) \\ (2) \\ (3)-2(2) \end{matrix}$$

$$[I]_\beta^\alpha = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

(b) Compute $[T]_\alpha^\beta$.

$$[T(v_1)]_\beta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad [T(\vec{v}_2)]_\beta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad [T(v_3)]_\beta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[T]_\alpha^\beta = \begin{bmatrix} | & | & | \\ [T(\vec{v}_1)]_\beta & [T(\vec{v}_2)]_\beta & [T(\vec{v}_3)]_\beta \\ | & | & | \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) Compute $[T]_\alpha^\alpha$.

$$[T]_\alpha^\alpha = [I]_\beta^\alpha [T]_\alpha^\beta = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

2. *(18 pts)* Find the matrices P and D in the diagonalization $D = P^{-1}AP$ of the matrix A below.

$$A = \begin{pmatrix} 2 & 7 \\ 0 & 3 \end{pmatrix}$$

$p(\lambda) = \det\begin{pmatrix} 2-\lambda & 7 \\ 0 & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda) \Rightarrow$ eigenvalues are 2,3

$\underline{\lambda=2:}$ $A-\lambda I = \begin{pmatrix} 0 & 7 \\ 0 & 1 \end{pmatrix}$ rank $= \dim(CS) = 1$

$\Rightarrow \dim(NS) = 2-1 = 1$

$NS = k\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so eigenvector $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\underline{\lambda=3:}$ $A-\lambda I = \begin{pmatrix} -1 & 7 \\ 0 & 0 \end{pmatrix}$ rank $= \dim(RS) = 1$

$\Rightarrow \dim(NS) = 2-1 = 1$

$NS = k\begin{pmatrix} 7 \\ 1 \end{pmatrix}$, so eigenvector $= \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

So $P = \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}$ eigenvectors

and $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ eigenvalues

3. *(12 pts)* Show that the Hermitian dot product on $\mathbb{C}^2$ satisfies the property below.

$$(4) \quad \langle \vec{v}, \vec{v} \rangle_H \geq 0, \text{ with equality if and only if } \vec{v} = \vec{0}$$

(Hint: Write $\vec{v} = (a + bi, c + di)$ and expand.)

$$\langle \vec{v}, \vec{v} \rangle_H = \left\langle \begin{pmatrix} a+bi \\ c+di \end{pmatrix}, \begin{pmatrix} a+bi \\ c+di \end{pmatrix} \right\rangle_H$$

$$= (a+bi)\overline{(a+bi)} + (c+di)\overline{(c+di)}$$

$$= (a+bi)(a-bi) + (c+di)(c-di)$$

$$= a^2+b^2 \quad + \quad c^2+d^2$$

This is a sum of squares of real numbers, so it must be ≥ 0.

And it can only be zero if $a=b=c=d=0$, in which case

$$\vec{v} = \begin{pmatrix} a+bi \\ c+di \end{pmatrix} = \begin{pmatrix} 0+0i \\ 0+0i \end{pmatrix} = \vec{0}$$

4. *(18 pts)*

(a) Show that the vectors $\vec{v}_1 = \left(\frac{-1}{7}, \frac{4}{7}, 0\right)$, $\vec{v}_2 = (0,0,1)$, $\vec{v}_3 = \left(\frac{2}{7}, \frac{-1}{7}, 0\right)$ form an orthonormal basis with respect to the inner product below.

$$\langle \vec{v}, \vec{w} \rangle = (A\vec{v}) \cdot (A\vec{w}) \quad \text{with} \quad A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 4 & 1 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} \langle \vec{v}_1, \vec{v}_2 \rangle &= (A\vec{v}_1)\cdot(A\vec{v}_2) = \begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} = 0 \\ \langle \vec{v}_1, \vec{v}_3 \rangle &= (A\vec{v}_1)\cdot(A\vec{v}_3) = \begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = 0 \\ \langle \vec{v}_2, \vec{v}_3 \rangle &= (A\vec{v}_2)\cdot(A\vec{v}_3) = \begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = 0 \end{aligned}\right\} \Rightarrow \text{orthogonal}$$

$$\left.\begin{aligned} \|\vec{v}_1\| &= \sqrt{\langle \vec{v}_1, \vec{v}_1 \rangle} = \sqrt{(A\vec{v}_1)\cdot(A\vec{v}_1)} = \sqrt{\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}} = 1 \\ \|\vec{v}_2\| &= \sqrt{\langle \vec{v}_2, \vec{v}_2 \rangle} = \sqrt{(A\vec{v}_2)\cdot(A\vec{v}_2)} = \sqrt{\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}} = 1 \\ \|\vec{v}_3\| &= \sqrt{\langle \vec{v}_3, \vec{v}_3 \rangle} = \sqrt{(A\vec{v}_3)\cdot(A\vec{v}_3)} = \sqrt{\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}} = 1 \end{aligned}\right\} \Rightarrow \text{normality}$$

(b) Use this orthonormality to find the coordinates of $\vec{x} = (1,2,3)$ with respect to this basis.

Being an orthonormal basis, coordinates can be computed with projections.
So with $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$, we have

$$c_1 = \langle \vec{x}, \vec{v}_1 \rangle = (A\vec{x})\cdot(A\vec{v}_1) = \begin{pmatrix}5\\3\\6\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} = 5$$

$$c_2 = \langle \vec{x}, \vec{v}_2 \rangle = (A\vec{x})\cdot(A\vec{v}_2) = \begin{pmatrix}5\\3\\6\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} = 3$$

$$c_3 = \langle \vec{x}, \vec{v}_3 \rangle = (A\vec{x})\cdot(A\vec{v}_3) = \begin{pmatrix}5\\3\\6\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = 6$$

5. *(14 pts)* The following information is given about the C^2 functions f, g, and h.

$$\begin{aligned}
\int_0^2 f^2\,dx &= 1 & \int_0^2 g^2\,dx &= 5 & \int_0^2 h^2\,dx &= 3 \\
\int_1^3 4f\,dx &= 3 & \int_1^3 4g\,dx &= 3 & \int_1^3 4h\,dx &= 4 \\
f'(0)+f(1) &= 0 & g'(0)+g(1) &= 1 & h'(0)+h(1) &= 1 \\
f'(2)/f(2) &= 7 & g'(2)/g(2) &= 1 & h'(2)/h(2) &= 6 \\
f''(3) &= 1 & g''(3) &= 2 & h''(3) &= 3
\end{aligned}$$

Is it possible to determine from this information if this trio of functions is linearly independent? If yes, then do so and explain your reasoning; if not, explain why not.

The function $T(y) = \begin{pmatrix} \int_1^3 4y\,dx \\ y'(0)+y(1) \\ y''(3) \end{pmatrix}$ is a linear transformation from C^2 to $\mathbb{R}^3$.

This gives us the Wronskian-like construction

$$W(f,g,h) = \det\begin{pmatrix} | & | & | \\ T(f) & T(g) & T(h) \\ | & | & | \end{pmatrix}$$

For these three functions f, g, h, we have

$$W(f,g,h) = \det\begin{pmatrix} 3 & 3 & 4 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= (1)\det\begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix} - (1)\det\begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix} = 2$$

This is nonzero, so $\{f, g, h\}$ is linearly independent.

6. *(18 pts)*

(a) Suppose that $M = R^{-1}DR$, where D is diagonal. Derive a decoupled system of equations in the variable $\vec{w}$ whose solutions would allow for solving for $\vec{y}$ in the system $\vec{y}' = M\vec{y}$, and give an explicit formula for $\vec{y}$ in terms of $\vec{w}$.

$$\vec{y}' = M\vec{y}$$

$$\vec{y}' = (R^{-1}DR)\vec{y}$$

$$R\vec{y}' = D\,R\vec{y}$$

Choosing $\vec{w} = R\vec{y}$, this becomes

$$\vec{w}' = D\vec{w}$$

This is a decoupled system because D is diagonal.

And $\vec{w} = R\vec{y} \Rightarrow \vec{y} = R^{-1}\vec{w}$

(b) Find a fundamental set of solutions to the system $\vec{v}' = A\vec{v}$, with

these matrices are inverses, so this is a conjugation.

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -2 & -2 & 3 \\ 4 & 3 & -5 \end{pmatrix}$$

eigenvalues are these diagonal entries.

eigenvectors are these columns.

$$[T]_{\mathcal{V}}^{\mathcal{V}} = [I]_{\mathcal{S}}^{\mathcal{V}}\,[T]_{\mathcal{S}}^{\mathcal{S}}\,[I]_{\mathcal{V}}^{\mathcal{S}}$$

So a f.s.s. is

$$\left\{ e^{3x}\begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix},\ e^{5x}\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix},\ e^{4x}\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} \right\}$$