EXAM 3

Math 216, 2017-2018 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name ______________________________

Disc.: Number__________ TA____________________ Day/Time____________________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ____________________

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

Total Score __________ (/100 points)

1. *(20 pts)* We consider here the vector space V, the bases $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\beta = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, and linear transformation $T : V \to V$ with

$$\begin{aligned} \vec{v}_1 &= 1\vec{w}_1 + 1\vec{w}_2 + 1\vec{w}_3 \\ \vec{v}_2 &= 0\vec{w}_1 + 1\vec{w}_2 + 2\vec{w}_3 \\ \vec{v}_3 &= 0\vec{w}_1 + 0\vec{w}_2 + 1\vec{w}_3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} T(\vec{v}_1) &= \vec{w}_1 \\ T(\vec{v}_2) &= \vec{w}_2 \\ T(\vec{v}_3) &= \vec{w}_3 \end{aligned}$$

(a) Find $[I]_\alpha^\beta$ and $[I]_\beta^\alpha$.

(b) Compute $[T]_\alpha^\beta$.

(c) Compute $[T]_\alpha^\alpha$.

2. *(18 pts)* Find the matrices P and D in the diagonalization $D = P^{-1}AP$ of the matrix A below.

$$A = \begin{pmatrix} 2 & 7 \\ 0 & 3 \end{pmatrix}$$

3. *(12 pts)* Show that the Hermitian dot product on $\mathbb{C}^2$ satisfies the property below.

$$(4) \quad \langle \vec{v}, \vec{v} \rangle_H \geq 0, \text{ with equality if and only if } \vec{v} = \vec{0}$$

(Hint: Write $\vec{v} = (a + bi, c + di)$ and expand.)

4. *(18 pts)*

(a) Show that the vectors $\vec{v}_1 = \left(\frac{-1}{7}, \frac{4}{7}, 0\right)$, $\vec{v}_2 = (0,0,1)$, $\vec{v}_3 = \left(\frac{2}{7}, \frac{-1}{7}, 0\right)$ form an orthonormal basis with respect to the inner product below.

$$\langle \vec{v}, \vec{w} \rangle = (A\vec{v}) \cdot (A\vec{w}) \quad \text{with} \quad A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 4 & 1 & 0 \end{pmatrix}$$

(b) Use this orthonormality to find the coordinates of $\vec{x} = (1,2,3)$ with respect to this basis.

5. *(14 pts)* The following information is given about the C^2 functions f, g, and h.

$$\begin{aligned} \int_0^2 f^2\,dx &= 1 & \int_0^2 g^2\,dx &= 5 & \int_0^2 h^2\,dx &= 3 \\ \int_1^3 4f\,dx &= 3 & \int_1^3 4g\,dx &= 3 & \int_1^3 4h\,dx &= 4 \\ f'(0)+f(1) &= 0 & g'(0)+g(1) &= 1 & h'(0)+h(1) &= 1 \\ f'(2)/f(2) &= 7 & g'(2)/g(2) &= 1 & h'(2)/h(2) &= 6 \\ f''(3) &= 1 & g''(3) &= 2 & h''(3) &= 3 \end{aligned}$$

Is it possible to determine from this information if this trio of functions is linearly independent? If yes, then do so and explain your reasoning; if not, explain why not.

6. *(18 pts)*

(a) Suppose that $M = R^{-1}DR$, where D is diagonal. Derive a decoupled system of equations in the variable $\vec{w}$ whose solutions would allow for solving for $\vec{y}$ in the system $\vec{y}' = M\vec{y}$, and give an explicit formula for $\vec{y}$ in terms of $\vec{w}$.

(b) Find a fundamental set of solutions to the system $\vec{v}' = A\vec{v}$, with

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -2 & -2 & 3 \\ 4 & 3 & -5 \end{pmatrix}$$