EXAM 2

Math 216, 2017-2018 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________________ Day/Time________________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ________________

1. ________
2. ________
3. ________
4. ________
5. ________
6. ________

Total Score ________ (/100 points)

1. *(16 pts)*

(a) Use the Wronskian to decide if the list below is linearly independent or linearly dependent.

$$x^2 - 1,\ \ 3x + 2,\ \ 2x^2 + 3x$$

$$W(x) = \det \begin{pmatrix} x^2-1 & 3x+2 & 2x^2+3x \\ 2x & 3 & 4x+3 \\ 2 & 0 & 4 \end{pmatrix}$$

$$= 2\big((3x+2)(4x+3) - (3)(2x^2+3x)\big)$$
$$+ 4\big((x^2-1)(3) - (2x)(3x+2)\big)$$
$$= 0$$

These functions are all analytic and $W(x) = 0$, so the list is linearly dependent.

(b) Use any technique from this course to decide if the list below is linearly independent or linearly dependent.

$3x^2e^{x^3} + 2\sin x\cos^2 x + 2\ln(3 + \sin x),\ 1x^2e^{x^3} + 3\sin x\cos^2 x + 2\ln(3 + \sin x),$
$6x^2e^{x^3} + 8\sin x\cos^2 x + 1\ln(3 + \sin x),\ 5x^2e^{x^3} + 1\sin x\cos^2 x + 5\ln(3 + \sin x)$

There are 4 vectors in this list, all of which are in

$$\operatorname{span}\left(x^2e^{x^3},\ \sin x\cos^2 x,\ \ln(3+\sin x)\right)$$

which is at most 3-dimensional.

So the list must be linearly dependent.

2. *(18 pts)* Find a fundamental set of real solutions to the differential equation $L(y) = 0$ whose characteristic polynomial is

$$p(\lambda) = (\lambda+2)^3(\lambda^2+5)(\lambda^2+6\lambda+10)^2(\lambda^3+4\lambda^2+5\lambda+2)$$

$\lambda^2+6\lambda+10 = (\lambda+3)^2+1$ has roots $\lambda = -3 \pm i$.

$\lambda^3+4\lambda^2+5\lambda+2$ has possible rat'l roots $\pm 1, \pm 2$; note that -1 is a root, so $\lambda+1$ is a factor. And

$$\begin{array}{r|l} & \lambda^2 + 3\lambda + 2 \\ \lambda+1 & \lambda^3 + 4\lambda^2 + 5\lambda + 2 \\ & \underline{\lambda^3 + \lambda^2} \\ & \quad 3\lambda^2 + 5\lambda + 2 \\ & \quad \underline{3\lambda^2 + 3\lambda} \\ & \qquad 2\lambda + 2 \\ & \qquad \underline{2\lambda + 2} \\ & \qquad\quad 0 \end{array}$$

$$\Rightarrow \lambda^3+4\lambda^2+5\lambda+2 = (\lambda+1)(\lambda^2+3\lambda+2) = (\lambda+1)^2(\lambda+2)$$

So

$$p(\lambda) = (\lambda+2)^3(\lambda^2+5)(\lambda-(-3+i))^2(\lambda-(-3-i))^2(\lambda+1)^2(\lambda+2)$$

$$= (\lambda+2)^4(\lambda^2+5)(\lambda-(-3+i))^2(\lambda-(-3-i))^2(\lambda+1)^2$$

and thus by the theorem from class, a f.s.s. is

$$\{e^{-2x}, xe^{-2x}, x^2e^{-2x}, x^3e^{-2x}, \cos(x\sqrt{5}), \sin(x\sqrt{5}),$$
$$e^{-3x}\cos x, e^{-3x}\sin x, xe^{-3x}\cos x, xe^{-3x}\sin x, e^{-x}, xe^{-x}\}$$

3. *(16 pts)* Find the form of a particular solution to the differential equation below. (You do not have to evaluate the unknown constants.)

$$y''' + 2y'' + 4y' + 8y = x\sin(2x) + \cos(2x)$$

Naive guess : $(c_0 + c_1 x)\cos(2x) + (d_0 + d_1 x)\sin(2x)$

$p(\lambda) = \lambda^3 + 2\lambda^2 + 4\lambda + 8$, and $r = 2i$;

$p(r) = p(2i) = -8i - 8 + 8i + 8 = 0$

$\Rightarrow$ $\pm 2i$ are factors

$\Rightarrow$ if m were ≥ 2 then degree of p would have to be ≥ 4 ;

so $m = 1$.

So our actual guess is

$$y_p = x^1(c_0 + c_1 x)\cos(2x) + x^1(d_0 + d_1 x)\sin(2x)$$

4. *(16 pts)* Find a particular solution to the differential equation below.

$$y' + y = e^{-x} \cos 3x$$

Noting that $e^{-x}\cos 3x = \mathrm{Re}\left(e^{(-1+3i)x}\right)$, we consider instead

$$z' + z = e^{(-1+3i)x}$$

Choose $z = Te^{(-1+3i)x}$. Then the equation becomes

$$T(-1+3i)\,e^{(-1+3i)x} + Te^{(-1+3i)x} = e^{(-1+3i)x}$$

$$T\left((-1+3i)+1\right) = 1$$

$$\Rightarrow T = \frac{1}{3i}$$

$$= -\frac{1}{3} i$$

$$\Rightarrow z = -\frac{1}{3} i\, e^{(-1+3i)x}$$

$$= -\frac{1}{3} i \left(e^{-x}\cos 3x + i e^{-x}\sin 3x\right)$$

$$= \left(\frac{1}{3}e^{-x}\sin 3x\right) + i\left(-\frac{1}{3}e^{-x}\cos 3x\right)$$

Then $y = \mathrm{Re}(z)$

$$= \frac{1}{3} e^{-x}\sin 3x$$

5. *(16 pts)* Bob says that the function $S : C^\infty \to \mathbb{R}$ defined below acts linearly. Is he right? If he is, prove it; if he is not, find an explicit counterexample.

$$S(f) = 3f'(0) + 2f(0) + 1$$

Bob is wrong.

Consider $g \in C^\infty$ defined by $g(x) = 0$; note $g'(x) = 0$. Then

$$S(2g) = 3\cdot 0 + 2\cdot 0 + 1 = 1$$

$$2S(g) = 2(3\cdot 0 + 2\cdot 0 + 1) = 2$$

So $S(2g) \neq 2S(g)$, so S is not linear.

6. *(18 pts)*

(a) Show that $(D-3)(x^k e^{3x}) = kx^{k-1}e^{3x}$ when $k \geq 1$.

$$(D-3)(x^k e^{3x}) = D(x^k e^{3x}) - 3(x^k e^{3x})$$

$$= (kx^{k-1})(e^{3x}) + (x^k)(3e^{3x}) - 3(x^k e^{3x})$$

$$= kx^{k-1}e^{3x}$$

(b) The differential operator $L : C^\infty \to C^\infty$ is defined by

$$L = (D-3)^3(D-1)(D^2 - 4D + 6)$$

Without citing theorems from section 4.2, show directly that x^2e^{3x} is in $\ker(L)$.

$$L(x^2e^{3x}) = (D-3)^3(D-1)(D^2-4D-6)(x^2e^{3x})$$

$$= (D-1)(D^2-4D-6)(D-3)^3(x^2e^{3x})$$

$$= (D-1)(D^2-4D-6)(D-3)(D-3)^2(x^2e^{3x})$$

$$= (D-1)(D^2-4D-6)(D-3)(2e^{3x})$$ (by (a))

$$= (D-1)(D^2-4D-6)(6e^{3x}-6e^{3x})$$

$$= (D-1)(D^2-4D-6)(0)$$

$$= 0$$

(c) The differential operator $Q : C^\infty \to C^\infty$ is defined by

$$Q = (D-1)(D^2 - 4D + 6)$$

(Note the relationship between this Q and the operator L on the previous page.) The function $4x^5e^{3x}$ is a solution to the differential equation $Q(y) = g(x)$.

Without citing theorems from section 4.3, find a solution to the differential equation $L(y) = g(x)$ of the form Ax^8e^{3x}.

$$L(Ax^8e^{3x}) = g(x)$$

$$(D-3)^3(Q)(Ax^8e^{3x}) = g(x)$$

$$(Q)(D-3)^3(Ax^8e^{3x}) = g(x)$$

$$(Q)(A\cdot(8\cdot7\cdot6)x^5e^{3x}) = g(x)$$

by (a)

We know that $4x^5e^{3x}$ solves $Q(y) = g(x)$, so we choose $A\cdot 8\cdot7\cdot6\, x^5e^{3x} = 4x^5e^{3x}$

$$A\cdot 8\cdot 7\cdot 6 = 4$$

$$A = \frac{1}{84}$$

So $y = \frac{1}{84}x^8e^{3x}$ solves $L(y) = g(x)$.