EXAM 1

Math 216, 2017-2018 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________________ Day/Time________________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ________________

1. ________
2. ________
3. ________
4. ________
5. ________
6. ________

Total Score ________ (/100 points)

1. *(21 pts)* We consider the matrix A below.

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

(a) Find a matrix M for which $MA = R$ is the reduced row echelon form of A.

$$(A \mid I) \rightsquigarrow \left(\begin{array}{cccc|ccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} 1 \\ 2 \\ 3 - 1 \end{array}$$

$$\left(\begin{array}{cccc|ccc} 1 & 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & -2 & 1 \end{array}\right) \begin{array}{l} 1 - 2 \\ 2 \\ 3 - 2\,2 \end{array}$$

$$\left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 4 & 3 & 3 & -2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & -2 & 1 \end{array}\right) \begin{array}{l} 1 - 2\,3 \\ 2 \\ 3 \end{array}$$

$$\underbrace{}_{R} \qquad \underbrace{}_{M}$$

(b) Find the complete set of solutions to the system $A\vec{x} = \vec{b}$, with $\vec{b} = (1, 2, 0)$.

$$A\vec{x} = \vec{b} \iff R\vec{x} = M\vec{b}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 9 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & -5 \end{array}\right) \begin{array}{l} \rightsquigarrow x_1 = 9 - 4x_4 \\ \rightsquigarrow x_2 = 2 - x_4 \\ \rightsquigarrow x_3 = -5 + 2x_4 \end{array} \Bigg\} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 - 4x_4 \\ 2 - 1x_4 \\ -5 + 2x_4 \\ 0 + 1x_4 \end{pmatrix}$$

$$\underbrace{}_{R} \qquad \underbrace{}_{M\vec{b}}$$

(c) Compute the determinant of M *without* using a cofactor expansion.

Row reduction uses only row operations that don't change determinant.

So $\det M = \det I = 1$.

2. *(16 pts)* Bob is doing a row reduction of a matrix with three rows, and in the interests of saving time is combining row operations – but is worried he might possibly be combining too many at a time. At one point, he contemplates doing a step as indicated below.

$$\left.\begin{array}{c} \\ \\ \\ \end{array}\right) \begin{array}{c} 3(1) - 2(3) \\ 5(2) + (1) + (3) \\ 2(2) - 4(3) \end{array}$$

Can you help Bob decide if this actually is a combination of row operations?

$$\underbrace{\begin{pmatrix} 3 & 0 & -2 \\ 1 & 5 & 1 \\ 0 & 2 & -4 \end{pmatrix}}_{E} \begin{pmatrix} \\ M \\ \\ \end{pmatrix} = \begin{pmatrix} \\ EM \\ \\ \end{pmatrix} \begin{array}{l} 3(1) - 2(3) \\ 5(2) + (1) + (3) \\ 2(2) - 4(3) \end{array}$$

$$\det E = (3)(-22) - (0)(-4) + (-2)(2) = -70$$

$\det E \neq 0$, so E is nonsingular and thus a product of elementary matrices. So Bob's step is a combination of row operations.

3. *(16 pts)* We consider the matrix B below.

$$B = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 6 & 0 \\ 2 & 5 & 4 \end{pmatrix}$$

(a) Compute the determinant of B using a cofactor expansion.

Along the third column:

$$\det B = 0 + 0 + (4)(6) = 24$$

(b) Compute the determinant of B *without* using a cofactor expansion, instead using antisymmetry and an established theorem about triangular matrices.

switching rows 1 & 2: $\begin{pmatrix} 0 & 6 & 0 \\ 1 & 3 & 0 \\ 2 & 5 & 4 \end{pmatrix}$

switching cols 1 & 2: $\begin{pmatrix} 6 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 2 & 4 \end{pmatrix}$

This is lower triangular and thus its determinant is the product of the diagonal entries, $6 \cdot 1 \cdot 4 = 24$

So $\det B = (-1)(-1)(24) = 24$.

4. *(15 pts)* Let D be the set of all diagonal 3×3 matrices with non-negative entries, with addition $\oplus$ and scalar multiplication $\otimes$ defined by

$$A \oplus B = AB$$

$$c \otimes \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = \begin{pmatrix} a_1^c & 0 & 0 \\ 0 & a_2^c & 0 \\ 0 & 0 & a_3^c \end{pmatrix}$$

(a) Show that D does satisfy the following condition: "There is a zero vector, 0, with $0 + v = v$ for all $v \in V$."

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in D$, and

$I \oplus A = IA = A$ for all $A \in D$.

So I is the zero vector in D, and the condition is satisfied.

(b) Show that D does satisfy the following condition: "$c(u + v) = cu + cv$ for all $u, v \in V$, $c \in \mathbb{R}$."

$$c \otimes (A \oplus B) = c \otimes \begin{pmatrix} a_1b_1 & 0 & 0 \\ 0 & a_2b_2 & 0 \\ 0 & 0 & a_3b_3 \end{pmatrix} = \begin{pmatrix} (a_1b_1)^c & 0 & 0 \\ 0 & (a_2b_2)^c & 0 \\ 0 & 0 & (a_3b_3)^c \end{pmatrix} = \begin{pmatrix} a_1^cb_1^c & 0 & 0 \\ 0 & a_2^cb_2^c & 0 \\ 0 & 0 & a_3^cb_3^c \end{pmatrix}$$

$$\|$$

$$(c \otimes A) \oplus (c \otimes B) = \begin{pmatrix} a_1^c & 0 & 0 \\ 0 & a_2^c & 0 \\ 0 & 0 & a_3^c \end{pmatrix} \oplus \begin{pmatrix} b_1^c & 0 & 0 \\ 0 & b_2^c & 0 \\ 0 & 0 & b_3^c \end{pmatrix} = \begin{pmatrix} a_1^c & 0 & 0 \\ 0 & a_2^c & 0 \\ 0 & 0 & a_3^c \end{pmatrix} \begin{pmatrix} b_1^c & 0 & 0 \\ 0 & b_2^c & 0 \\ 0 & 0 & b_3^c \end{pmatrix}$$

(c) Explain fully why D is not a vector space.

D does not satisfy the requirement of elements having additive inverses. An additive inverse of $A \in D$ by $\oplus$ must be the multiplicative inverse A^{-1} – but $A \in D$ with a zero on the diagonal are not invertible.

So D is <u>not</u> a vector space.

5. *(16 pts)* Let P be the collection of polynomials with roots at $x = 0$ and $x = 1$ (and possibly elsewhere), with the usual addition and scalar multiplication. Is P a vector space? Prove or disprove.

We will show that P is closed under addition and scalar multiplication.

① Suppose $f, g \in P$. Then f, g have roots at 0 and 1.

$$\Rightarrow \begin{cases} f(x) = q_1(x)(x-0)(x-1) \\ g(x) = q_2(x)(x-0)(x-1) \end{cases}$$

$$\Rightarrow f(x) + g(x) = (q_1(x) + q_2(x))(x-0)(x-1)$$

$\Rightarrow$ $f+g$ has roots at 0 and 1.

$\Rightarrow$ $f+g \in P$.

So P is closed under addition.

② Suppose $f \in P$. Then f has roots at 0 and 1.

$$\Rightarrow f(x) = q(x)(x-0)(x-1)$$

$$\Rightarrow (cf)(x) = (cq(x))(x-0)(x-1)$$

$\Rightarrow$ cf has roots at 0 and 1.

$\Rightarrow$ $cf \in P$.

So P is closed under scalar multiplication.

By ① and ② P is a subspace of the known vector space F, so P is a vector space.

6. *(16 pts)* The functions f, g, and h are known to be linearly independent. Decide if the list $f-g$, $g-h$, $f+2g+3h$ is linearly independent or linearly dependent.

A relation between $u=f-g$, $v=g-h$, and $w=f+2g+3h$ would be

$$c_1 u + c_2 v + c_3 w = 0$$

Being independent, $\beta = \{f,g,h\}$ is a basis for $V = \operatorname{span}\{f,g,h\}$, and we can write the above relation in coordinates as

$$c_1[u]_\beta + c_2[v]_\beta + c_3[w]_\beta = [0]_\beta$$

$$c_1\begin{pmatrix}1\\-1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\-1\end{pmatrix} + c_3\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 1\\-1 & 1 & 2\\0 & -1 & 3\end{pmatrix}\vec{c} = \vec{0}$$

This matrix has $\det = 6 \neq 0$, so the trivial solution is the only solution. Thus there is no significant relation, and $\{u, v, w\}$ is linearly independent.