EXAM 3

Math 216, 2016-2017 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number______ TA______ Day/Time______

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______

Total Score ______ (/100 points)

1. *(20 pts)* T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, and we have

$$[T]_\mathcal{S}^\mathcal{S} = \begin{pmatrix}3 & 1\\ 2 & 4\end{pmatrix} \quad \text{and} \quad \mathcal{V} = \{v_1, v_2\} \quad \text{and} \quad [v_1]_\mathcal{S} = \begin{pmatrix}5\\2\end{pmatrix},\ [v_2]_\mathcal{S} = \begin{pmatrix}7\\3\end{pmatrix},\ [x]_\mathcal{S} = \begin{pmatrix}3\\1\end{pmatrix}$$

(a) Compute $[T(x)]_\mathcal{S}$

$$[T(x)]_\mathcal{S} = [T]_\mathcal{S}^\mathcal{S}[x]_\mathcal{S} = \begin{pmatrix}3 & 1\\ 2 & 4\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}10\\10\end{pmatrix}$$

(b) Compute $[I]_\mathcal{V}^\mathcal{S}$ and $[I]_\mathcal{S}^\mathcal{V}$.

$$[I]_\mathcal{V}^\mathcal{S} = \left([\vec{v}_1]_\mathcal{S}\ [\vec{v}_2]_\mathcal{S}\right) = \begin{pmatrix}5 & 7\\ 2 & 3\end{pmatrix}$$

$$[I]_\mathcal{S}^\mathcal{V} = \left([I]_\mathcal{V}^\mathcal{S}\right)^{-1} = \begin{pmatrix}3 & -7\\ -2 & 5\end{pmatrix}\Big/(5\cdot3-7\cdot2) = \begin{pmatrix}3 & -7\\ -2 & 5\end{pmatrix}$$

(c) Compute $[T]_\mathcal{V}^\mathcal{V}[x]_\mathcal{V}$ WITHOUT computing either $[T]_\mathcal{V}^\mathcal{V}$ or $[x]_\mathcal{V}$.

$$[T]_\mathcal{V}^\mathcal{V}[x]_\mathcal{V} = [T(x)]_\mathcal{V} = [I]_\mathcal{S}^\mathcal{V}[T(x)]_\mathcal{S}$$

$$= \begin{pmatrix}3 & -7\\ -2 & 5\end{pmatrix}\begin{pmatrix}10\\10\end{pmatrix} = \begin{pmatrix}-40\\30\end{pmatrix}$$

(d) Compute $[T]_\mathcal{V}^\mathcal{V}$.

$$[T]_\mathcal{V}^\mathcal{V} = [I]_\mathcal{S}^\mathcal{V}[T]_\mathcal{S}^\mathcal{S}[I]_\mathcal{V}^\mathcal{S}$$

$$= \begin{pmatrix}3 & -7\\ -2 & 5\end{pmatrix}\begin{pmatrix}3 & 1\\ 2 & 4\end{pmatrix}\begin{pmatrix}5 & 7\\ 2 & 3\end{pmatrix}$$

$$= \begin{pmatrix}-5 & -25\\ 4 & 18\end{pmatrix}\begin{pmatrix}5 & 7\\ 2 & 3\end{pmatrix} = \begin{pmatrix}-75 & -110\\ 56 & 82\end{pmatrix}$$

2. *(18 pts)* The matrix A is given below.

$$A = \begin{pmatrix} -8 & 6 \\ -15 & 11 \end{pmatrix}$$

(a) Find the eigenvalue(s) for A.

$$p(\lambda) = \det\begin{pmatrix} -8-\lambda & 6 \\ -15 & 11-\lambda \end{pmatrix} = (-88 - 3\lambda + \lambda^2) - (-90)$$
$$= \lambda^2 - 3\lambda + 2$$
$$= (\lambda-1)(\lambda-2)$$

eigenvalues are roots : $\lambda_1 = 1$, $\lambda_2 = 2$

(b) Find the eigenvector(s) for the above eigenvalue(s).

$\lambda_1 = 1$: $NS\underbrace{\begin{pmatrix} -9 & 6 \\ -15 & 10 \end{pmatrix}}_{\text{rank}=1}$ has dim $= 2-1 = 1$; $NS = \left\{k\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$ $\vec{v}_1$, is an eigenvector

$\lambda_2 = 2$: $NS\underbrace{\begin{pmatrix} -10 & 6 \\ -15 & 9 \end{pmatrix}}_{\text{rank}=1}$ has dim $= 2-1 = 1$; $NS = \left\{k\begin{pmatrix} 3 \\ 5 \end{pmatrix}\right\}$ $\vec{v}_2$, is an eigenvector

(c) Find the matrices D and M that diagonalize A by $D = MAM^{-1}$.

With $A = [T]_{\mathcal{S}}^{\mathcal{S}}$, $\mathcal{V} = \{\vec{v}_1, \vec{v}_2\}$, A is diagonalized by

$$D = [T]_{\mathcal{V}}^{\mathcal{V}} = [I]_{\mathcal{S}}^{\mathcal{V}}[T]_{\mathcal{S}}^{\mathcal{S}}[I]_{\mathcal{V}}^{\mathcal{S}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$ eigenvalues

$$= M \quad A \quad M^{-1}$$

So $M = \left([I]_{\mathcal{V}}^{\mathcal{S}}\right)^{-1} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} \Big/ (2\cdot5 - 3\cdot3) = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$

3. *(18 pts)* Let V be the inner product space made with the vector space $\mathbb{R}^2$ and the inner product below, and consider the vectors $\vec{a}$ and $\vec{b}$ below.

$$< \vec{v}, \vec{w} >= A\vec{v} \cdot A\vec{w}, \quad \text{with } A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad \vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a) Find the angle between $\vec{a}$ and $\vec{b}$ in V.

$$\|\vec{a}\| = \sqrt{\langle \vec{a}, \vec{a} \rangle} = \sqrt{A\vec{a} \cdot A\vec{a}} = \sqrt{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = 1$$

$$\|\vec{b}\| = \sqrt{\langle \vec{b}, \vec{b} \rangle} = \sqrt{A\vec{b} \cdot A\vec{b}} = \sqrt{\begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix}} = \sqrt{10}$$

$$\langle \vec{a}, \vec{b} \rangle = A\vec{a} \cdot A\vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3$$

$$\Theta = \arccos\left(\frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\| \, \|\vec{b}\|}\right) = \arccos\left(\frac{3}{1 \cdot \sqrt{10}}\right) = \arccos\left(\frac{3}{\sqrt{10}}\right)$$

(b) Apply the Gram-Schmidt procedure to the basis $\{\vec{a}, \vec{b}\}$ to find an orthonormal basis for V.

$$\vec{v}_1 = \frac{\vec{a}}{\|\vec{a}\|} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} / 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{x}_2 = \vec{b} - \langle \vec{b}, \vec{v}_1 \rangle \vec{v}_1 = \vec{b} - \langle \vec{b}, \vec{a} \rangle \vec{a}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - (3)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\|\vec{x}_2\| = \sqrt{\langle \vec{x}_2, \vec{x}_2 \rangle} = \sqrt{A\vec{x}_2 \cdot A\vec{x}_2} = \sqrt{\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}} = 1$$

$$\vec{v}_2 = \frac{\vec{x}_2}{\|\vec{x}_2\|} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} / 1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

The resulting orthonormal basis is $\{\vec{v}_1, \vec{v}_2\} = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix}\right\}$.

4. *(16 pts)*

(a) Compute the Hermitian dot product of $\vec{a} = (1-3i, 2+i)$ and $\vec{b} = (4+i, 3-2i)$.

$$\left\langle \begin{pmatrix} 1-3i \\ 2+i \end{pmatrix}, \begin{pmatrix} 4+i \\ 3-2i \end{pmatrix} \right\rangle_H = \begin{pmatrix} 1-3i \\ 2+i \end{pmatrix} \cdot \begin{pmatrix} 4-i \\ 3+2i \end{pmatrix}$$

$$= (4 - 12i - i - 3) + (6 + 3i + 4i - 2)$$

$$= 5 - 6i$$

(b) Given the matrix M below,

$$M = \begin{pmatrix} 2-3i & 7-5i \\ 3+4i & 5+i \end{pmatrix}$$

find a matrix N for which the statement

$$< M\vec{v}, \vec{w} >_H = < \vec{v}, N\vec{w} >_H$$

is true for all vectors $\vec{v}, \vec{w} \in \mathbb{C}^2$.

$$N = M^* = \overline{M}^T = \begin{pmatrix} 2+3i & 7+5i \\ 3-4i & 5-i \end{pmatrix}^T$$

$$= \begin{pmatrix} 2+3i & 3-4i \\ 7+5i & 5-i \end{pmatrix}$$

(c) Find the unique value of c that makes the matrix U below unitary.

$$U = \begin{pmatrix} 1-i & 1-i \\ 1+i & c \end{pmatrix} /2$$

Need the columns to be (Hermitian) orthogonal, so

$$0 = \left\langle \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}, \begin{pmatrix} 1-i \\ c \end{pmatrix} \right\rangle = \begin{pmatrix} 1-i \\ 1+i \end{pmatrix} \cdot \begin{pmatrix} 1+i \\ \bar{c} \end{pmatrix} = 2 + \bar{c}(1+i)$$

$$\Rightarrow \bar{c} = \frac{-2}{1+i} = \frac{-2(1-i)}{(1+i)(1-i)} = \frac{-2(1-i)}{2} = -1+i$$

$$\Rightarrow c = -1-i$$

5. *(10 pts)* Consider the vector space V of all continuous functions $f : \mathbb{R}^2 \to \mathbb{R}^1$, and the linear transformation $T : V \to \mathbb{R}^3$ given by $T(f) = (f(0,0), f(1,0), f(0,1))$. Use a Wronskian based on T to decide if the three functions below are linearly independent or linearly dependent.

$$\begin{aligned} f_1(x,y) &= x - 3y + 2 \\ f_2(x,y) &= 2x + y - 4 \\ f_3(x,y) &= 3x - 5y - 2 \end{aligned}$$

$$W = \det\begin{pmatrix} T(f_1) & T(f_2) & T(f_3) \end{pmatrix}$$

$$= \det\begin{pmatrix} 2 & -4 & -2 \\ 3 & -2 & 1 \\ -1 & -3 & -7 \end{pmatrix}$$

$$= (2)(17) - (-4)(-20) + (-2)(-11)$$

$$= 34 - 80 + 22$$

$$= -24$$

$W(f_1, f_2, f_3) \neq 0$, so this list is linearly independent.

6. *(18 pts)* Find a fundamental set of solutions to the system below.

$$\begin{aligned} y_1' &= -6y_1 + 6y_2 \\ y_2' &= -12y_1 + 11y_2 \end{aligned}$$

(Hint: Consider how the vectors $(2,3)$ *and* $(3,4)$ *might be useful.)*

$\vec{y}' = A\vec{y}$, with $A = \begin{pmatrix} -6 & 6 \\ -12 & 11 \end{pmatrix}$.

$A\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 3 \end{pmatrix} \Longrightarrow \vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 3$.

$A\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 4 \end{pmatrix} \Longrightarrow \vec{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = 2$.

By a theorem from class, a f.s.s. is

$$\left\{ e^{\lambda_1 x}\vec{v}_1, \; e^{\lambda_2 x}\vec{v}_2 \right\}$$

$$= \left\{ e^{3x}\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \; e^{2x}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$$