EXAM 2

Math 216, 2016-2017 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number_________ TA_________________ Day/Time_________________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: _________________

1. _________
2. _________
3. _________
4. _________
5. _________
6. _________

Total Score _________ (/100 points)

1. *(16 pts)* We consider here the collection S of all functions of the form

$$ae^{x+b} + ce^{3x+d}$$

(a) Show that S is a vector space.

We will show that S is a subspace of C^0.

① Closed under addition:

$$\left(a_1e^{x+b_1} + c_1e^{3x+d_1}\right) + \left(a_2e^{x+b_2} + c_2e^{3x+d_2}\right)$$

$$= \left(a_1e^{b_1}e^{x} + c_1e^{d_1}e^{3x}\right) + \left(a_2e^{b_2}e^{x} + c_2e^{d_2}e^{3x}\right)$$

$$= \left(a_1e^{b_1} + a_2e^{b_2}\right)e^{x} + \left(c_1e^{d_1} + c_2e^{d_2}\right)e^{3x}$$

This is of the required form, so S is closed under addition.

② Closed under scalar multiplication:

$$k\left(ae^{x+b} + ce^{3x+d}\right) = (ka)e^{x+b} + (kc)e^{3x+d}$$

This is of the required form, so S is closed under scalar multiplication.

Thus S is a subspace of C^0, and thus it is a vector space.

(b) Find the dimension of S.

$$ae^{x+b} + ce^{3x+d} = \left(ae^{b}\right)e^{x} + \left(ce^{d}\right)e^{3x}$$

So $S = \text{span}\{e^x, e^{3x}\}$.

$$W(x) = \det\begin{pmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{pmatrix} = 2e^{4x}$$, which is not identically zero,

so e^x, e^{3x} are linearly independent and thus form a basis for $S = \text{span}\{e^x, e^{3x}\}$.

So $\dim(S) = 2$.

2. *(18 pts)* Consider the following arguments that Bob makes using the Wronskian. In each case:

I. if his conclusions are invalid, explain why they are invalid, and if possible what might fix them.

II. if his conclusions are valid, identify what parts of his arguments (if any) are unnecessary.

(a) For the functions $f_1, \ldots, f_4$, Bob computes correctly that $w(x) = \sin x$. He also notes correctly that these functions are analytic. From these facts he concludes that the list of functions is linearly independent.

This is a valid conclusion, but his observation that the functions are analytic is unnecessary.

(b) For the functions $g_1, \ldots, g_6$, Bob computes correctly that $w(x) = 0$. He also notes correctly that these functions are analytic. From these facts he concludes that the list of functions is linearly dependent.

This is a valid conclusion, and nothing in his argument is unnecessary.

(c) For the functions $p_1, \ldots, p_5$ (all of which are degree 4 polynomials), Bob notes correctly that all degree 4 polynomials can be written as linear combinations of them. He also notes correctly that they are all solutions to a linear differential equation $L(y) = 0$ whose coefficient functions are continuous. He then concludes (i) that he does not yet have enough information to decide if this list of functions is linearly independent, (ii) that he should thus proceed to compute the Wronskian, and (iii) that if he finds a value x_0 with $w(x_0) = 0$ then he could conclude the list was linearly dependent.

(i) is wrong. Because $p_1, \ldots, p_5$ span P_4, a 5-dim vector space, we know the list is linearly independent.

(ii) is wrong. Given (i), there is no need to compute the Wronskian.

(iii) is wrong. To draw that conclusion, he would need to know also that the differential equation was of order 5, and that the lead coefficient function was never zero.

(Of course that is not possible, because we already know the list is linearly independent.

3. *(16 pts)* Find a fundamental set of real solutions to the differential equation whose partially factored characteristic polynomial is below.

$$p(\lambda) = (\lambda - 4)^3(\lambda^3 + 4\lambda^2 + \lambda - 6)(\lambda - (2 + 3i))^2(\lambda - (2 - 3i))^2$$

Possible rational roots of $q(x) = \lambda^3 + 4\lambda^2 + \lambda - 6$ are: $\pm 1, \pm 2, \pm 3, \pm 6$.

$q(1) = 0 \Rightarrow 1$ is a root $\Rightarrow (\lambda - 1)$ is a factor.

$$\begin{array}{r|l} & \lambda^2 + 5\lambda + 6 \\ \lambda - 1 & \lambda^3 + 4\lambda^2 + \lambda - 6 \\ & \underline{\lambda^3 - \lambda^2} \\ & \quad 5\lambda^2 + \lambda - 6 \\ & \quad \underline{5\lambda^2 - 5\lambda} \\ & \qquad 6\lambda - 6 \\ & \qquad \underline{6\lambda - 6} \\ & \qquad\quad 0 \end{array}$$

$\Rightarrow q(\lambda) = (\lambda - 1)(\lambda^2 + 5\lambda + 6)$
$= (\lambda - 1)(\lambda + 2)(\lambda + 3)$

So $p(\lambda) = (\lambda - 4)^3(\lambda - 1)(\lambda + 2)(\lambda + 3)(\lambda - (2 + 3i))^2(\lambda - (2 - 3i))^2$

So a f.s.s. is:

$\{e^{4x}, xe^{4x}, x^2e^{4x}, e^x, e^{-2x}, e^{-3x}, e^{2x}\cos 3x, e^{2x}\sin 3x,$
$xe^{2x}\cos 3x, xe^{2x}\sin 3x\}$

4. *(16 pts)* Find a particular solution to the differential equation below.

$$y' + 3y = \sin 2x$$

Guess: $y = A\cos 2x + B\sin 2x$

$y' = -2A\sin 2x + 2B\cos 2x$

$\Rightarrow \left(-2A\sin 2x + 2B\cos 2x\right) + 3\left(A\cos 2x + B\sin 2x\right) = \sin 2x$

$\left(-2A + 3B\right)\sin 2x + \left(2B + 3A\right)\cos 2x = (1)\sin 2x + (0)\cos 2x$

$\Rightarrow \begin{array}{l} -2A + 3B = 1 \\ 3A + 2B = 0 \end{array} \Rightarrow \begin{array}{l} A = -2/13 \\ B = 3/13 \end{array}$

$\Rightarrow y_p = -\frac{2}{13}\cos 2x + \frac{3}{13}\sin 2x$

5. *(16 pts)* In a chosen set of units, the motion of a mechanical system is modeled by the differential equation

$$y'' + fy' + 9y = 4\cos 2t$$

Find an expression for the gain in the system in terms of the friction coefficient f. (Compute this directly as the amplitude of the output divided by the amplitude of the input – do not cite shortcuts from examples from the notes.) What value of f maximizes the gain?

We consider the associated complex equation:

$$z'' + fz' + 9z = 4e^{2it}$$

Guess: $z = T \cdot 4e^{2it}$. Equation becomes

$$(-4 + 2if + 9) T \cdot 4e^{2it} = 4e^{2it}$$

$$\Rightarrow T = \frac{1}{5 + 2fi}$$

$$= \underbrace{\frac{1}{\sqrt{25 + 4f^2}}}_{G = \text{magnitude of } T} e^{-i\phi} = G e^{-i\phi}$$

So the solution to the complex equation is

$$z = T \cdot 4e^{2it}$$

$$= Ge^{-i\phi} \cdot 4e^{2it}$$

$$= 4Ge^{(2t-\phi)i}$$

So the solution to the real equation is

$$y = \mathrm{Re}(z)$$

$$= 4G\cos(2t - \phi)$$

and thus the gain is

$$\text{gain} = \frac{4G}{4} = G = \frac{1}{\sqrt{25 + 4f^2}}$$

This is maximized when the denominator is as small as possible, so $f = 0$.

6. *(18 pts)* Consider the four dimensional vector space $V = \text{span}\{\sin x, \cos x, e^{-x}\sin x, e^{-x}\cos x\}$, and the linear transformation $T : V \to C^0$ given by

$$T(f) = f'' + 2f' + 2f$$

(a) Find the images of the four given vectors spanning V.

$$T(\sin x) = (-\sin x) + 2(\cos x) + 2(\sin x) = \sin x + 2\cos x$$

$$T(\cos x) = (-\cos x) + 2(-\sin x) + 2(\cos x) = \cos x - 2\sin x$$

$$T(e^{-x}\sin x) = (e^{-x}(-2\cos x)) + 2(e^{-x}(-\sin x + \cos x)) + 2(e^{-x}\sin x) = 0$$

$$T(e^{-x}\cos x) = (e^{-x}(2\sin x)) + 2(e^{-x}(-\sin x - \cos x)) + 2(e^{-x}\cos x) = 0$$

(b) Use the result of part (a) to compute the dimension of $\text{im}(T)$.

The image of T is the span of the four vectors above, which is the span of the two vectors

$$\sin x + 2\cos x, \quad \cos x - 2\sin x$$

Neither of these is a multiple of the other, so these vectors are independent, and thus are a basis for $\text{im}(T)$.

So $\dim(\text{im}(T)) = 2$

(c) Use the result of part (b) to compute the dimension of $\ker(T)$.

The domain is given to be four dimensional, so

$$\dim(\ker(T)) + \dim(\text{im}(T)) = 4$$

So $\dim(\ker(T)) + 2 = 4$

and thus $\dim(\ker(T)) = 2$