EXAM 1

Math 216, 2016-2017 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA____________ Day/Time____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________

1. ________
2. ________
3. ________
4. ________
5. ________
6. ________

Total Score ________ (/100 points)

1. *(16 pts)* Find the complete set of solutions to the system below.

$$\begin{aligned} x - 3z &= -2 \\ 2y + z &= 5 \\ x + 2y - 2z &= 3 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 2 & 1 & 5 \\ 1 & 2 & -2 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & 1 & 5 \end{array}\right)\begin{array}{l} 1 \\ 2 \\ 3 - 1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{array}{l} 1 \\ 2 \\ 3 - 2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 1/2 & 5/2 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{array}{l} 1 \\ 2/2 \\ 3 \end{array}$$

$$\underbrace{}_{\text{RREF}}$$

$x = -2 + 3z$

$y = 5/2 - \frac{1}{2}z$

So: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 + 3z \\ 5/2 - \frac{1}{2}z \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 5/2 \\ 0 \end{pmatrix} + z\begin{pmatrix} 3 \\ -1/2 \\ 1 \end{pmatrix}$

2. *(16 pts)* Find the unique values of c and k that make the system below have infinitely many solutions (x, y, z). (As always, don't forget to explain your reasoning!)

$$\begin{aligned} 2x + y + cz &= k \\ x + y + 5z &= 5 \\ y + 2z &= 4 \end{aligned}$$

(Hint: Find c first, and then make convenient use of the following nonsingular matrix:)

$$\begin{pmatrix} 0 & 1 & -1 \\ -2 & 4 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

In order for it to be possible to have ∞'ly many solutions, the coefficient matrix must be singular. So

$$0 = \det \begin{pmatrix} 2 & 1 & c \\ 1 & 1 & 5 \\ 0 & 1 & 2 \end{pmatrix} = 2(-3) - 1(2-c)$$

$$= c - 8 \qquad \Rightarrow \boxed{c = 8}$$

So the augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 1 & 8 & k \\ 1 & 1 & 5 & 5 \\ 0 & 1 & 2 & 4 \end{array}\right)$$

Since at this point it would be convenient to have a row reduction matrix E, we consider the possibility that the matrix in the hint might serve. Using it as such, the system becomes

$$E \left(\begin{array}{ccc|c} 2 & 1 & 8 & k \\ 1 & 1 & 5 & 5 \\ 0 & 1 & 2 & 4 \end{array}\right) = \begin{pmatrix} 0 & 1 & -1 \\ -2 & 4 & -1 \\ 1 & -2 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 2 & 1 & 8 & k \\ 1 & 1 & 5 & 5 \\ 0 & 1 & 2 & 4 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2k+16 \\ 0 & 0 & 0 & k-6 \end{array}\right)$$

In order for a solution (and thus ∞'ly many solutions) to exist, we must have $\boxed{k = 6}$.

3. *(18 pts)* The 3×3 matrix A is row reduced to the identity matrix by the following sequence of row operations:

    i. The first row is multiplied by 2.
    ii. 4 times the first row is added to the second row.
    iii. The second row is divided by 3.
    iv. 2 times the second row is added to the third row.
    v. The third row is multiplied by 5.

(a) Compute the determinant of A.

The effect of the operations above on the determinant is summarized by

$$(5)(1)(\tfrac{1}{3})(1)(2)(\det A) = \det I = 1$$

So $\frac{10}{3} \det A = 1$ and thus $\det A = \frac{3}{10}$.

(b) Find elementary matrices $E_1, \ldots, E_5$ such that $A = E_1 E_2 E_3 E_4 E_5$.

The row reduction above in matrix form is

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}}_{F_5} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}}_{F_4} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{F_3} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{F_2} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{F_1} A = I$$

Then $A = F_1^{-1} F_2^{-1} F_3^{-1} F_4^{-1} F_5^{-1}$

$= E_1 E_2 E_3 E_4 E_5$

So we can use

$$E_1 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \quad E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{pmatrix}$$

4. *(16 pts)* Bob says he likes to live dangerously when it comes to "combining row operations". At one point in a row reduction, he takes the following "step":

$$\left.\begin{array}{l} 3① + 2② \\ 2② - 3① + 4③ \\ 4③ + 2① \end{array}\right)$$

(a) Is Bob's above "step" equivalent to left multiplication by some matrix? If so, what is the matrix?

Yes, the matrix is

$$F = \begin{pmatrix} 3 & 2 & 0 \\ -3 & 2 & 4 \\ 2 & 0 & 4 \end{pmatrix}$$

(b) Is Bob's "step" actually a combination of row operations? (If so, you are not required to identify the row operations; but in either case you must justify your reasoning.)

$$\det F = 3(8) - 2(-20) + 0(-4)$$
$$= 64 \neq 0$$
$$\Rightarrow \text{rref}(F) = I$$

$\Rightarrow$ F is a product of elementary matrices

$\Rightarrow$ Bob's "step" is a combination of row operations.

5. *(16 pts)* The following determinants are given (and easily checked).

$$\det\begin{pmatrix}233 & 1 & 234\\ 234 & 0 & 235\\ 233 & 0 & 234\end{pmatrix} = -1 \qquad \det\begin{pmatrix}233 & 0 & 234\\ 234 & 1 & 235\\ 233 & 0 & 234\end{pmatrix} = 0 \qquad \det\begin{pmatrix}233 & 0 & 234\\ 234 & 0 & 235\\ 233 & 1 & 234\end{pmatrix} = 1$$

Show how you can use this information to compute the determinant below.

$$\det\begin{pmatrix}233 & 126 & 234\\ 234 & 511 & 235\\ 233 & 376 & 234\end{pmatrix}$$

Using multilinearity in the second column, we can write

$$\det\begin{pmatrix}233 & 126 & 234\\ 234 & 511 & 235\\ 233 & 376 & 234\end{pmatrix} = 126 \det\begin{pmatrix}233 & 1 & 234\\ 234 & 0 & 235\\ 233 & 0 & 234\end{pmatrix} + 511 \det\begin{pmatrix}233 & 0 & 234\\ 234 & 1 & 235\\ 233 & 0 & 234\end{pmatrix} + 376 \det\begin{pmatrix}233 & 0 & 234\\ 234 & 0 & 235\\ 233 & 1 & 234\end{pmatrix}$$

$$= 126(-1) + 511(0) + 376(1)$$

$$= 250$$

6. *(18 pts)* We are interesed in the question of whether the list of vectors below is linearly independent or linearly dependent.

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 6 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 5 \\ 6 \end{pmatrix}$$

(a) Show a step-by-step argument, starting from the definition of linear dependence, that results in a matrix equation whose solution(s) would allow for resolving the question.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is l.d. $\iff$ $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ has a significant solution.

$$\iff c_1\begin{pmatrix} 2 \\ 1 \\ 3 \\ 6 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 2 \\ 5 \\ 6 \end{pmatrix} = \vec{0} \quad \text{has a significant solution.}$$

$$\iff \begin{array}{l} 2c_1 + 1c_2 + 0c_3 = 0 \\ 1c_1 + 3c_2 + 2c_3 = 0 \\ 3c_1 + 1c_2 + 5c_3 = 0 \\ 6c_1 + 2c_2 + 6c_3 = 0 \end{array} \quad \text{has a significant solution.}$$

$$\iff \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 3 & 1 & 5 & 0 \\ 6 & 2 & 6 & 0 \end{array}\right) \quad \text{has a significant solution.}$$

(b) Suppose you knew the reduced row echelon form R for the matrix above. What requirement would R have to satisfy to allow you to conclude the list to be dependent? Why? (Note, you do not have to find R or decide if R actually satisfies this requirement.)

If R were to have a column with no pivot, then

$\Rightarrow$ there is a free variable

$\Rightarrow$ given the trivial solution there must be ∞'ly many solutions

$\Rightarrow$ there must be a significant solution

$\Rightarrow$ the list is dependent