EXAM 3

Math 216, 2016-2017 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________________ Day/Time________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. ________
2. ________
3. ________
4. ________
5. ________
6. ________

Total Score ________ (/100 points)

1. *(15 pts)* We have bases $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\mathcal{W} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ for $\mathbb{R}^3$, with the vectors (in terms of the standard basis $\mathcal{S}$) given by

$$\vec{v}_1 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix}-1\\0\\0\end{pmatrix} \quad \vec{w}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix} \quad \vec{w}_2 = \begin{pmatrix}0\\2\\1\end{pmatrix} \quad \vec{w}_3 = \begin{pmatrix}2\\1\\0\end{pmatrix}$$

T is a linear transformation, and

$$T(\vec{v}_1) = 3\vec{w}_1, \quad T(\vec{v}_2) = 4\vec{w}_2, \quad T(\vec{v}_3) = 5\vec{w}_3$$

(a) Compute $[T]_{\mathcal{V}}^{\mathcal{W}}$, $[I]_{\mathcal{V}}^{\mathcal{S}}$, and $[I]_{\mathcal{W}}^{\mathcal{S}}$.

$$[T]_{\mathcal{V}}^{\mathcal{W}} = \begin{bmatrix} [T(\vec{v}_1)]_{\mathcal{W}} & [T(\vec{v}_2)]_{\mathcal{W}} & [T(\vec{v}_3)]_{\mathcal{W}} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$[I]_{\mathcal{V}}^{\mathcal{S}} = \begin{bmatrix} [\vec{v}_1]_{\mathcal{S}} & [\vec{v}_2]_{\mathcal{S}} & [\vec{v}_3]_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[I]_{\mathcal{W}}^{\mathcal{S}} = \begin{bmatrix} [\vec{w}_1]_{\mathcal{S}} & [\vec{w}_2]_{\mathcal{S}} & [\vec{w}_3]_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(b) Compute $[T]_{\mathcal{S}}^{\mathcal{S}}$.

$$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{W}}^{\mathcal{S}} [T]_{\mathcal{V}}^{\mathcal{W}} [I]_{\mathcal{S}}^{\mathcal{V}}$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 10 \\ 0 & 8 & 5 \\ 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & 3 & 0 \\ -5 & 0 & 8 \\ 0 & 3 & 4 \end{pmatrix}$$

To compute $[I]_{\mathcal{S}}^{\mathcal{V}}$ we invert $[I]_{\mathcal{V}}^{\mathcal{S}}$:

$$\left(\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right) \begin{array}{l} 2 \\ 3 \\ -1 \end{array}$$

$$\underbrace{\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{matrix}}_{[I]_{\mathcal{S}}^{\mathcal{V}}}$$

2. *(15 pts)* Find all of the eigenvalues and eigenvectors of the matrix A below.

$$A = \begin{pmatrix} 7 & -10 \\ 3 & -4 \end{pmatrix}$$

$$p(\lambda) = \det \begin{pmatrix} \lambda-7 & 10 \\ -3 & \lambda+4 \end{pmatrix} = (\lambda-7)(\lambda+4) + 30$$

$$= \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$$

Eigenvalues are 1, 2.

<u>For $\lambda_1 = 1$:</u>

$$\lambda I - A = \begin{pmatrix} -6 & 10 \\ -3 & 5 \end{pmatrix}$$ $\leftarrow$ rank $= 1 \Rightarrow \dim(NS) = 1$
so any vector in NS is a basis.

$\vec{v}_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ is the eigenvector.

<u>For $\lambda_2 = 2$:</u>

$$\lambda I - A = \begin{pmatrix} -5 & 10 \\ -3 & 6 \end{pmatrix}$$ $\leftarrow$ rank $= 1 \Rightarrow \dim(NS) = 1$
so any vector in NS is a basis.

$\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is the eigenvector.

3. *(15 pts)* Suppose that A is an invertible $n \times n$ matrix. Show that

$$\langle \vec{v}, \vec{w} \rangle = (A\vec{v}) \cdot (A\vec{w})$$

is an inner product on $\mathbb{R}^n$

We check the four conditions:

1. $\langle \vec{w}, \vec{v} \rangle = (A\vec{w}) \cdot (A\vec{v}) = (A\vec{v}) \cdot (A\vec{w}) = \langle \vec{v}, \vec{w} \rangle$ ✓

2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = (A(\vec{u}+\vec{v})) \cdot (A\vec{w}) = (A\vec{u} + A\vec{v}) \cdot (A\vec{w})$

   $= (A\vec{u}) \cdot (A\vec{w}) + (A\vec{v}) \cdot (A\vec{w}) = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ ✓

3. $\langle c\vec{v}, \vec{w} \rangle = (A(c\vec{v})) \cdot (A\vec{w}) = (cA\vec{v}) \cdot (A\vec{w}) = c(A\vec{v}) \cdot (A\vec{w})$

   $= c\langle \vec{v}, \vec{w} \rangle$ ✓

4. $\langle \vec{v}, \vec{v} \rangle = (A\vec{v}) \cdot (A\vec{v})$

   $= w_1^2 + w_2^2 + w_3^2$ where $\vec{w} = A\vec{v}$

   This is clearly ≥ 0.

   Also, if it is $= 0$, then $w_1 = w_2 = w_3 = 0$

   $\Rightarrow A\vec{v} = \vec{w} = \vec{0}$

   and since A is invertible, $A\vec{v} = \vec{0} \Rightarrow \vec{v} = \vec{0}$.

4. *(20 pts)* The 2×2 "Hessian matrix"

$$H = \begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix}$$

is the matrix of second derivatives that is used in the second derivative test in multivariable calculus. Recall that for functions f in C^2, we have $f_{xy} = f_{yx}$.

(a) Explain how you know that, for functions in C^2, the Hessian matrix must be orthogonally diagonalizable, with real eigenvalues λ_1 and λ_2.

For these functions, $f_{xy} = f_{yx}$, and so

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

is symmetric. And all real symmetric matrices are orthogonally diagonalizable, so $H = PDP^{-1}$ where P is orthogonal.

(b) The second order behavior of $f \in C^2$ near the point $\vec{a}$ is described by the expression $(\vec{x} - \vec{a}) \cdot H(\vec{x} - \vec{a})$. Use part (a) to show that this can be rewritten as $\lambda_1 z_1^2 + \lambda_2 z_2^2$, for some $\vec{z} = (z_1, z_2)$. *(Hint: Recall that orthogonal matrices preserve dot products.)*

$$= (\vec{x}-\vec{a}) \cdot (PDP^{-1})(\vec{x}-\vec{a})$$

Because P is orthogonal from part (a), so is P^{-1}. And orthogonal matrices preserve dot products, so we can rewrite this as

$$= P^{-1}(\vec{x}-\vec{a}) \cdot P^{-1}(PDP^{-1})(\vec{x}-\vec{a}) = P^{-1}(\vec{x}-\vec{a}) \cdot DP^{-1}(\vec{x}-\vec{a})$$

Choosing $\vec{z} = P^{-1}(\vec{x}-\vec{a})$, this becomes

$$= \vec{z} \cdot D\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \end{pmatrix} = \lambda_1 z_1^2 + \lambda_2 z_2^2$$

(c) In what way do the values z_1 and z_2 relate $(\vec{x} - \vec{a})$ to the eigenvectors of H?

$\vec{z} = P^{-1}(\vec{x}-\vec{a})$ means that $(\vec{x}-\vec{a}) = P\vec{z}$

The columns of P are the eigenvectors $\vec{v}_1, \vec{v}_2$, so

$$(\vec{x}-\vec{a}) = \begin{pmatrix} \uparrow & \uparrow \\ v_1 & v_2 \\ \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1\vec{v}_1 + z_2\vec{v}_2$$

So z_1, z_2 are the coordinates of $(\vec{x}-\vec{a})$ w.r.t. the eigenbasis $\mathcal{V} = \{\vec{v}_1, \vec{v}_2\}$.

5. *(15 pts)* The following arithmetic is given.

$$① \quad A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ \frac{-i}{2} & \frac{1}{2} \end{pmatrix} = BDC$$

$$② \quad I = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ \frac{-i}{2} & \frac{1}{2} \end{pmatrix} = BC$$

Find a real fundamental set of solutions to the system $\vec{y}' = A\vec{y}$.

② $\Rightarrow$ B, C are inverses of each other. So ① is a diagonalization, $A = PDP^{-1}$, and $P = B$.

So $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $1-i$, giving us the solution

$$e^{(1-i)x} \begin{pmatrix} -i \\ 1 \end{pmatrix} = e^x(\cos x - i\sin x)\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -e^x \sin x \\ e^x \cos x \end{pmatrix} + i \begin{pmatrix} -e^x \cos x \\ -e^x \sin x \end{pmatrix}$$

So a real fundamental set of solutions is

$$\left\{ \begin{pmatrix} -e^x \sin x \\ e^x \cos x \end{pmatrix}, \begin{pmatrix} -e^x \cos x \\ -e^x \sin x \end{pmatrix} \right\}$$

6. *(20 pts)* The matrix A has Jordan form $J = [T]_{\mathcal{V}}^{\mathcal{V}}$ by way of the Jordan basis $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, with

$$J = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad \vec{v}_1 = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

Find a fundamental set of solutions to the system $\vec{y}' = A\vec{y}$.

Using matrix exponentials with the given Jordan basis, we get solutions

$$e^{xA}\vec{V}_1 = e^{5x}\vec{V}_1 = e^{5x}\begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$$

$$e^{xA}\vec{V}_2 = e^{3x}\vec{V}_2 = e^{3x}\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$$

$$e^{xA}\vec{V}_3 = e^{3x}\left(\vec{V}_3 + x\vec{V}_2\right) = e^{3x}\left(\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + x\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}\right)$$

These form a fundamental set of solutions.