EXAM 3

Math 216, 2016-2017 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name ______________________________

Disc.: Number__________ TA____________________ Day/Time____________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________

Total Score __________ (/100 points)

1. *(15 pts)* We have bases $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\mathcal{W} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ for $\mathbb{R}^3$, with the vectors (in terms of the standard basis $\mathcal{S}$) given by

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \vec{w}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad \vec{w}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

T is a linear transformation, and

$$T(\vec{v}_1) = 3\vec{w}_1, \quad T(\vec{v}_2) = 4\vec{w}_2, \quad T(\vec{v}_3) = 5\vec{w}_3$$

(a) Compute $[T]_{\mathcal{V}}^{\mathcal{W}}$, $[I]_{\mathcal{V}}^{\mathcal{S}}$, and $[I]_{\mathcal{W}}^{\mathcal{S}}$.

(b) Compute $[T]_{\mathcal{S}}^{\mathcal{S}}$.

2. *(15 pts)* Find all of the eigenvalues and eigenvectors of the matrix A below.

$$A = \begin{pmatrix} 7 & -10 \\ 3 & -4 \end{pmatrix}$$

3. *(15 pts)* Suppose that A is an invertible $n \times n$ matrix. Show that

$$\langle \vec{v}, \vec{w} \rangle = (A\vec{v}) \cdot (A\vec{w})$$

is an inner product on $\mathbb{R}^n$

4. *(20 pts)* The 2×2 "Hessian matrix"

$$H = \begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix}$$

is the matrix of second derivatives that is used in the second derivative test in multivariable calculus. Recall that for functions f in C^2, we have $f_{xy} = f_{yx}$.

(a) Explain how you know that, for functions in C^2, the Hessian matrix must be orthogonally diagonalizable, with real eigenvalues λ_1 and λ_2.

(b) The second order behavior of $f \in C^2$ near the point $\vec{a}$ is described by the expression $(\vec{x} - \vec{a}) \cdot H(\vec{x} - \vec{a})$. Use part (a) to show that this can be rewritten as $\lambda_1 z_1^2 + \lambda_2 z_2^2$, for some $\vec{z} = (z_1, z_2)$. *(Hint: Recall that orthogonal matrices preserve dot products.)*

(c) In what way do the values z_1 and z_2 relate $(\vec{x} - \vec{a})$ to the eigenvectors of H?

5. *(15 pts)* The following arithmetic is given.

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ \frac{-i}{2} & \frac{1}{2} \end{pmatrix}$$

$$I = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ \frac{-i}{2} & \frac{1}{2} \end{pmatrix}$$

Find a real fundamental set of solutions to the system $\vec{y}\,' = A\vec{y}$.

6. *(20 pts)* The matrix A has Jordan form $J = [T]_{\mathcal{V}}^{\mathcal{V}}$ by way of the Jordan basis $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, with

$$J = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad \vec{v}_1 = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

Find a fundamental set of solutions to the system $\vec{y}\,' = A\vec{y}$.