EXAM 2

Math 216, 2016-2017 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number______ TA______ Day/Time______

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ______

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______

Total Score ______ (/100 points)

1. *(20 pts)*

(a) Use the Wronskian to decide if this list of functions is or is not linearly independent: $f_1(x) = 2x^2 + x + 4,\ f_2(x) = x^2 + 3x + 2,\ f_3(x) = 4x + 7,\ f_4(x) = -x^2 + 2x + 3.$

$$W(x) = \det\begin{pmatrix} 2x^2+x+4 & x^2+3x+2 & 4x+7 & -x^2+2x+3 \\ 4x+1 & 2x+3 & 4 & -2x+2 \\ 4 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$= 0$ (by cofactor expansion along bottom row.)

These functions are analytic because they are polynomials; so $W(x)=0 \Rightarrow$ this list is linearly dependent.

(b) Use a dimension argument to draw the same conclusion.

All four of these functions are in $\mathrm{span}\{1, x, x^2\}$, which is 3-dimensional. So we have four vectors in a 3-d vector space, so they must be linearly dependent.

2. *(20 pts)* Find a real fundamental set of solutions to the differential equation $L(y) = 0$ whose characteristic polynomial factors as

$$p(\lambda) = (\lambda - 6)^3(\lambda^3 - 6\lambda^2 + 11\lambda - 6)(\lambda - (4+5i))^2(\lambda - (4-5i))^2$$

Evident roots are: $r_1 = 6$ $(m_1=3)$, $r_2 = 4+5i$, $\bar{r}_2 = 4-5i$ $(m_2=2)$

To factor $q(x) = (\lambda^3 - 6\lambda^2 + 11\lambda - 6)$, we consider the possible rational roots, which are: $\pm1, \pm2, \pm3, \pm6$. $q(1) = 0$, so $(\lambda - 1)$ is a factor.

Dividing, we get

$$\begin{array}{r|l} & \lambda^2 - 5\lambda + 6 \\ \hline \lambda - 1 & \lambda^3 - 6\lambda^2 + 11\lambda - 6 \\ & \underline{\lambda^3 - \lambda^2} \\ & \quad -5\lambda^2 + 11\lambda - 6 \\ & \quad \underline{-5\lambda^2 + 5\lambda} \\ & \qquad 6\lambda - 6 \\ & \qquad \underline{6\lambda - 6} \\ & \qquad\quad 0 \checkmark \end{array}$$

$\Rightarrow q(x) = (\lambda - 1)(\lambda - 2)(\lambda - 3)$ and thus roots $r_3 = 1$, $r_4 = 2$, $r_5 = 3$, all with multiplicity equal to 1.

From these roots then, using the theorem from class, we get the following real f.s.s.:

$r_1 = 6$ $(m_1 = 3)$: $e^{6x}, xe^{6x}, x^2e^{6x}$

$r_2 = 4+5i$, $\bar{r}_2 = 4-5i$ $(m_2 = 2)$: $e^{4x}\cos 5x, e^{4x}\sin 5x, xe^{4x}\cos 5x, xe^{4x}\sin 5x$

$r_3 = 1$ $(m=1)$: e^{1x}

$r_4 = 2$ $(m=1)$: e^{2x}

$r_5 = 3$ $(m=1)$: e^{3x}

3. *(20 pts)* Find a particular solution to the differential equation below.

$$y' - 4y = xe^{4x}$$

Naive guess : $(c_1x+c_0)e^{4x}$.

$p(\lambda) = \lambda - 4$ has root 4.

"Magic number" of RHS is $r = 4+0i = 4$

r is a root of p, with m=1

So the form of a particular solution is

$$y_p = x^1(c_1x+c_0)e^{4x}$$
$$= (c_1x^2+c_0x)e^{4x}$$

Then

$$y_p' = (2c_1x+c_0)e^{4x} + (c_1x^2+c_0x)(4e^{4x})$$
$$= (4c_1x^2 + (2c_1+4c_0)x + c_0)e^{4x}$$

and the differential equation becomes

$$(\cancel{4c_1x^2} + (2c_1+\cancel{4c_0})x + c_0)e^{4x} - 4(\cancel{c_1x^2}+\cancel{c_0x})e^{4x} = xe^{4x}$$

$$((2c_1)x + (c_0))e^{4x} = ((1)x+(0))e^{4x}$$

$$\Rightarrow \left.\begin{array}{l} 2c_1 = 1 \\ c_0 = 0 \end{array}\right\} \Rightarrow c_1 = \frac{1}{2},\ c_0 = 0$$

So a particular solution is

$$y_p = \frac{1}{2}x^2e^{4x}$$

4. *(20 pts)* The input $h(t)$ and output $y(t)$ of a given system are related by the differential equation below.

$$y'' - 2y' + 5y = h(t)$$

Suppose that the input frequency is $f = \omega/2\pi = 2/2\pi$, as represented by the input function $h(t) = \sin(2t)$. Compute the resulting sinusoidal output function, and determine from that output the gain and the phase shift in the system at that frequency.

The associated complex equation to $y''-2y'+5y = \sin 2t$ is

$$z''-2z'+5z = e^{2it}$$

We guess

$$z = Te^{2it}$$

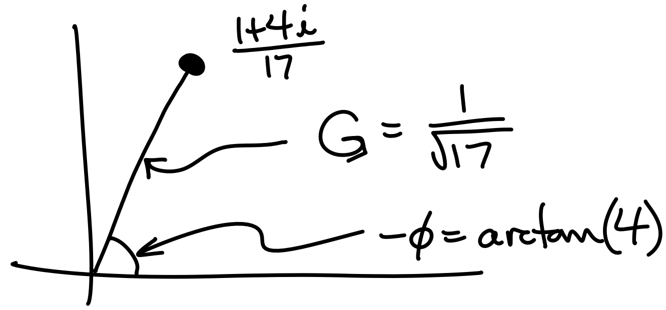
and the equation becomes

$$-4Te^{2it} - 4iTe^{2it} + 5Te^{2it} = e^{2it}$$

$$(-4-4i+5)Te^{2it} = e^{2it}$$

$$\Rightarrow T = \frac{1}{1-4i} = \frac{1+4i}{17}$$

$$= Ge^{-\phi i}$$



Then $z = Te^{2it}$

$$= Ge^{-\phi i}e^{2it} = Ge^{i(2t-\phi)}$$

So $y = \mathrm{Im}(z)$

$$= G\sin(2t-\phi)$$

Compared to the input function $h(t) = \sin 2t$, this has gain $= G = \frac{1}{\sqrt{17}}$

and phase shift $= \phi = -\arctan(4)$

5. *(10 pts)* Show that $T : C^1[0,1] \to \mathbb{R}$, defined by the formula below, is a linear transformation.

$$T(f) = \int_0^1 e^x f'(x)\, dx$$

$$T(c_1f_1+c_2f_2) = \int_0^1 e^x (c_1f_1+c_2f_2)'(x)\, dx$$

$$= \int_0^1 e^x \left(c_1f_1'(x) + c_2f_2'(x)\right) dx$$

$$= \int_0^1 e^x c_1f_1'(x) + e^x c_2f_2'(x)\, dx$$

$$= \int_0^1 e^x c_1f_1'(x)\, dx + \int_0^1 e^x c_2f_2'(x)\, dx$$

$$= c_1\int_0^1 e^x f_1'(x)\, dx + c_2\int_0^1 e^x f_2'(x)\, dx$$

$$= c_1T(f_1) + c_2T(f_2)$$

So T is a linear transformation.

6. *(10 pts)*

(a) Compute $(D-3)x^k e^{3x}$, where k is a positive integer.

$$(D-3)x^k e^{3x} = D\left(x^k e^{3x}\right) - 3\left(x^k e^{3x}\right)$$
$$= \left(kx^{k-1}e^{3x} + \cancel{x^k 3e^{3x}}\right) - \cancel{\left(3x^k e^{3x}\right)}$$
$$= kx^{k-1}e^{3x}$$

(b) Compute $(D-3)e^{3x}$.

$$(D-3)e^{3x} = D\left(e^{3x}\right) - 3\left(e^{3x}\right)$$
$$= 3e^{3x} - 3e^{3x}$$
$$= 0$$

(c) Use the results above to show that x^2e^{3x} is in the kernel of $L = (D-3)^4(D-2)^3$.

$$L\left(x^2e^{3x}\right) = (D-3)^4(D-2)^3\left(x^2e^{3x}\right)$$
$$= (D-2)^3(D-3)(D-3)\underbrace{(D-3)^2\left(x^2e^{3x}\right)}_{\text{(by part (a))}}$$
$$= (D-2)^3(D-3)\underbrace{(D-3)\left(2\cdot 1\cdot e^{3x}\right)}_{\text{(by part (b))}}$$
$$= (D-2)^3(D-3)(0)$$
$$= 0$$

So x^2e^{3x} is in the kernel of L.