EXAM 1

Math 216, 2016-2017 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions ____________

Disc.: Number____________ TA________________________ Day/Time________________________

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ________________________

Total Score ____________ (/100 points)

1. *(16 pts)* Bob says he likes to live dangerously when it comes to "combining row operations". At one point in a row reduction, he takes the following "step":

$$\left.\begin{array}{l} 3① + 2③ \\ 1① - 1② + 1③ \\ 1② + 1③ \end{array}\right.$$

(a) Is Bob's above "step" equivalent to left multiplication by some matrix? If so, what is the matrix?

Yes: $F = \begin{pmatrix} 3 & 0 & 2 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

(b) Is Bob's "step" actually a combination of row operations? (If so, you are not required to identify the row operations; but in either case you must justify your reasoning.)

$\det F = -4 \neq 0$

$\Rightarrow \operatorname{rref}(F) = I$

$\Rightarrow$ F is a product of elementary matrices

$\Rightarrow$ Bob's "step" is a combination of row operations

2. *(16 pts)* In a calculation, you find yourself with the equation $A\vec{x} = A\vec{y}$, where

$$A = \begin{pmatrix} 3 & 6 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$$

(a) You would like to conclude that $\vec{x} = \vec{y}$, but your friend Bob says you cannot because: (1) that would require left multiplication by an inverse of A, yet (2) A is not invertible. Is Bob right, partly right, or all wrong? And, is it legitimate to deduce that $\vec{x} = \vec{y}$?

Bob is partly right; right that A is not invertible, but wrong in thinking this is required.

We can left cancel the A because $\operatorname{rref}(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has a pivot in every column, and thus A has the uniqueness property as required for left cancellation.

So it is legitimate to deduce that $\vec{x} = \vec{y}$.

(b) Find the complete set of solutions to the system $A\vec{x} = (3, -2, 2)$.

$$\left(\begin{array}{cc|c} 3 & 6 & 3 \\ 0 & 2 & -2 \\ 1 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 2 \end{array}\right) \begin{array}{l} ①/3 \\ ②/2 \\ ③ \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{array}\right) \begin{array}{l} ① \\ ② \\ ③ - ① \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right) \left.\begin{array}{l} ① - 2② \\ ② \\ ③ + ② \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = 3 \\ x_2 = -1 \end{array}$$

3. *(10 pts)* Suppose that M is a product of elementary matrices, $M = E_k E_{k-1} \cdots E_2 E_1$. Show directly that M is invertible.

Elementary matrices are all invertible, so we can write

$$(E_1^{-1} E_2^{-1} \cdots E_k^{-1}) M = (E_1^{-1} E_2^{-1} \cdots E_k^{-1})(E_k \cdots E_2 E_1)$$

$$= I$$

So M is invertible.

4. *10 pts)* The columns of A are $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, and its determinant is 3. The columns of B are $5\vec{a}_1 + 2\vec{a}_3$, $\vec{a}_3$, $7\vec{a}_2$. Compute the determinant of B.

$$\det \begin{pmatrix} 5\vec{a}_1 + 2\vec{a}_3 & \vec{a}_3 & 7\vec{a}_2 \end{pmatrix}$$

$$= 5 \det \begin{pmatrix} \vec{a}_1 & \vec{a}_3 & 7\vec{a}_2 \end{pmatrix} + \underbrace{2 \det \begin{pmatrix} \vec{a}_3 & \vec{a}_3 & 7\vec{a}_2 \end{pmatrix}}_{= 0 \text{ by antisymmetry}}$$

$$= 35 \det \begin{pmatrix} \vec{a}_1 & \vec{a}_3 & \vec{a}_2 \end{pmatrix}$$

$$= -35 \det \begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{pmatrix}$$

$$= (-35)(3) = -105$$

5. *(16 pts)* Consider the list of vectors below.

$$\vec{v}_1 = \begin{pmatrix} 12 \\ 5 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 6 \\ 2 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$$

(a) Write a single vector equation whose solution(s) would indicate whether this list of vectors is linearly independent or linearly dependent (you do not have to find the solution(s)), and identify what specifically about the solution(s) to that equation would indicate which was the case.

$$c_1 \begin{pmatrix} 12 \\ 5 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \\ 6 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

only the trivial solution $\Rightarrow$ l. i. (From the definition

significant solutions $\Rightarrow$ l.d. of l.i., l.d.)

(b) Write a matrix whose reduced row echelon form would serve a similar purpose (you do not have to find that reduced row echelon form), and identify how you would use the reduced row echelon form to conclude which was the case.

The above is equivalent to the matrix equation

$$\begin{pmatrix} 12 & 1 & 0 \\ 5 & 3 & 1 \\ 2 & 6 & 0 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Sols relate to rref of this matrix by

pivot in every col. $\Rightarrow$ unique sols $\Rightarrow$ l. i.

some col. w/o pivot $\Rightarrow$ multiple sols $\Rightarrow$ l.d.

6. *(16 pts)* The vectors $\vec{v}$ and $\vec{w}$ are fixed vectors in $\mathbb{R}^3$. S is the set

$$S = \{A \in M_{3\times 3}(\mathbb{R}) | (A\vec{v}) \cdot \vec{w} = 0\}$$

using the usual addition and scalar multiplication. Is S a vector space?

① Say $B, C \in S$. Then

$$(B\vec{v}) \cdot \vec{w} = 0 \quad , \quad (C\vec{v}) \cdot \vec{w} = 0$$

$$\text{So } ((B+C)\vec{v}) \cdot \vec{w} = (B\vec{v} + C\vec{v}) \cdot \vec{w}$$

$$= (B\vec{v}) \cdot \vec{w} + (C\vec{v}) \cdot \vec{w}$$

$$= 0 + 0 = 0$$

So $B+C \in S$. Thus S is closed under addition.

② Say $B \in S$. Then

$$(B\vec{v}) \cdot \vec{w} = 0$$

$$\text{So } ((kB)\vec{v}) \cdot \vec{w} = (k(B\vec{v})) \cdot \vec{w}$$

$$= (k)((B\vec{v}) \cdot \vec{w}) = (k)(0) = 0$$

So $kB \in S$. Thus S is closed under scalar-vector mult.

By ① + ②, S is a subspace of $M_{3\times 3}(\mathbb{R})$.

So S is a vector space

7. *(16 pts)* Let $\mathcal{V}$ be the basis of P_3 (the vector space of polynomials that are at most cubic) consisting of the vectors $x^3 - 2x - 3$, $6x + 1$, $4x^2 + 5x$, $8x^3 - 7x + 9$. The vector w is the polynomial $4x^3 - 9$. The coordinates $[w]_{\mathcal{V}}$ can be written in the form $M^{-1}\vec{b}$, for some matrix M and some real vector $\vec{b}$. Find M and $\vec{b}$.

Suppose $[w]_{\mathcal{V}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$. Then

$$c_1\left(x^3 - 2x - 3\right) + c_2\left(6x+1\right) + c_3\left(4x^2 + 5x\right) + c_4\left(8x^3 - 7x + 9\right) = 4x^3 - 9$$

$$\left(c_1 + 8c_4\right)x^3 + \left(4c_3\right)x^2 + \left(-2c_1 + 6c_2 + 5c_3 - 7c_4\right)x + \left(-3c_1 + c_2 + 9c_4\right) = 4x^3 - 9$$

$$\begin{aligned} c_1 \qquad\qquad\qquad + 8c_4 &= 4 \\ 4c_3 \qquad &= 0 \\ -2c_1 + 6c_2 + 5c_3 - 7c_4 &= 0 \\ -3c_1 + 1c_2 \qquad + 9c_4 &= -9 \end{aligned}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 8 \\ 0 & 0 & 4 & 0 \\ -2 & 6 & 5 & -7 \\ -3 & 1 & 0 & 9 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}}_{[w]_{\mathcal{V}}} = \underbrace{\begin{pmatrix} 4 \\ 0 \\ 0 \\ -9 \end{pmatrix}}_{\vec{b}}$$

As required,

$$[w]_{\mathcal{V}} = M^{-1}\vec{b}$$