EXAM 3

Math 216, 2015-2016 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________________ Day/Time________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. ________

2. ________

3. ________

4. ________

5. ________

Total Score ________ (/100 points)

1. *(20 pts)* In this problem we consider the inner product space $P = \text{span}\{\cos x, \sin x\} \subset C[0, 2\pi]$, using the L^2 inner product.

(a) Show that $\cos x$ is orthogonal to $\sin x$.

$$\langle \cos x, \sin x \rangle = \int_0^{2\pi} \cos x \sin x \, dx = \frac{1}{2} \sin^2 x \Big]_0^{2\pi} = 0$$

(b) Find the value k such that $\{k \cos x, k \sin x\}$ is an orthonormal basis for P.

$$1 = \langle k\cos x, k\cos x \rangle = \int_0^{2\pi} k^2 \cos^2 x \, dx = k^2 \int_0^{2\pi} \frac{1+\cos 2x}{2} \, dx = k^2 \pi$$

$$1 = \langle k\sin x, k\sin x \rangle = \int_0^{2\pi} k^2 \sin^2 x \, dx = k^2 \int_0^{2\pi} \frac{1-\cos 2x}{2} \, dx = k^2 \pi$$

$$\Rightarrow k = \frac{1}{\sqrt{\pi}}$$

(c) Suppose that $f = a \cos x + b \sin x$, and that the values of f are known but the values of a and b are not. Find formulas for a and b in terms of f. (Be sure to indicate clearly where you use part (b) in your argument.)

By (b), the coordinates of f equal the projections of f by these orthonormal basis vectors.

And $f = a\cos x + b\sin x = (a\sqrt{\pi})\left(\frac{1}{\sqrt{\pi}}\cos x\right) + (b\sqrt{\pi})\left(\frac{1}{\sqrt{\pi}}\sin x\right)$

So $a\sqrt{\pi} = \int_0^{2\pi} f(x)\left(\frac{1}{\sqrt{\pi}}\cos x\right) dx \Rightarrow a = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x \, dx$

$b\sqrt{\pi} = \int_0^{2\pi} f(x)\left(\frac{1}{\sqrt{\pi}}\sin x\right) dx \Rightarrow b = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x \, dx$

2. *(20 pts)*

(a) Diagonalize the matrix A below.

$$A = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}$$

$$p(\lambda) = \det\begin{pmatrix} -1-\lambda & 2 \\ -6 & 6-\lambda \end{pmatrix} = -(1+\lambda)(6-\lambda) + 12 = \lambda^2 - 5\lambda + 6$$

$$= (\lambda-2)(\lambda-3) \Rightarrow \lambda = 2, 3.$$

$A - 2I = \begin{pmatrix} -3 & 2 \\ -6 & 4 \end{pmatrix}$, $\dim(RS) = 1 \Rightarrow \dim(NS) = 1$, so only eigenvector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

$A - 3I = \begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix}$, $\dim(RS) = 1 \Rightarrow \dim(NS) = 1$, so only eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

So $\boxed{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}}$

(b) Find a fundamental set of solutions to the system $\vec{y}' = A\vec{y}$.

By the theorem from class, a f.s.s. is

$$\left\{ e^{2x}\begin{pmatrix} 2 \\ 3 \end{pmatrix}, e^{3x}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

3. *(20 pts)*

(a) The arithmetic below is given. Suppose that B represents a linear transformation T with respect to the standard basis $\mathcal{S}$. Find the change of basis matrix $[I]^{\mathcal{S}}_{\mathcal{V}}$, where $\mathcal{V}$ is a Jordan basis for T. (Be sure to clearly show your reasoning.)

$$I = \begin{pmatrix} 1 & -4 & 2 \\ 1 & -1 & 0 \\ -1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

$$B = \underbrace{\begin{pmatrix} -4 & -3 & -10 \\ -1 & 3 & -1 \\ 5 & 2 & 10 \end{pmatrix}}_{[T]^{\mathcal{S}}_{\mathcal{S}}} = \underbrace{\begin{pmatrix} 1 & -4 & 2 \\ 1 & -1 & 0 \\ -1 & 3 & -1 \end{pmatrix}}_{[I]^{\mathcal{S}}_{\mathcal{V}}} \underbrace{\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}}_{[T]^{\mathcal{V}}_{\mathcal{V}}} \underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}}_{[I]^{\mathcal{V}}_{\mathcal{S}}}$$

This is in Jordan form, so $\mathcal{V}$ is a Jordan basis. The change of basis matrices must be arranged like this for consistency, so $[I]^{\mathcal{S}}_{\mathcal{V}}$ is this matrix.

(b) There is a system $\vec{u}' = J\vec{u}$ (where J is in Jordan form) whose solutions relate to solutions to the system $\vec{y}' = B\vec{y}$. Find this system and the matrix C with $\vec{u} = C\vec{y}$, and show this relationship holds.

Writing the above arithmetic as $B = PJP^{-1}$, we can substitute to get

$$\vec{y}' = B\vec{y}$$

$$\vec{y}' = PJP^{-1}\vec{y}$$

$$(P^{-1}\vec{y})' = J(P^{-1}\vec{y})$$

Then $\vec{y}$ is a solution to $\vec{y}' = B\vec{y}$ if and only if $\vec{u} = P^{-1}\vec{y}$ is a solution to $\vec{u}' = J\vec{u}$.

So the matrix C is $P^{-1} = [I]^{\mathcal{V}}_{\mathcal{S}} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$

4. *(20 pts)* The only eigenvector for the matrix A below is the indicated vector $\vec{v}_1$ below.

$$[T]_{\mathcal{S}}^{\mathcal{S}} = \begin{pmatrix} 5 & -1 \\ 9 & -1 \end{pmatrix} \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Find a fundamental set of solutions to the system $\vec{y}' = A\vec{y}$.

Choose $\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and the basis $\mathcal{W} = \{\vec{v}_1, \vec{w}\}$. Then

$$[T]_{\mathcal{W}}^{\mathcal{W}} = [I]_{\mathcal{S}}^{\mathcal{W}} [T]_{\mathcal{S}}^{\mathcal{S}} [I]_{\mathcal{W}}^{\mathcal{S}}$$

$$= \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$$

Solve $\vec{z}' = [T]_{\mathcal{W}}^{\mathcal{W}} \vec{z} = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix} \vec{z}$

$$z_1' = 2z_1 - z_2$$

$$z_2' = 2z_2 \longrightarrow z = C_2 e^{2x}$$

$$z_1' = 2z_1 - C_2 e^{2x} \xrightarrow{\text{part.}} z_{1p} = Axe^{2x}$$

$$z_1' = 2z_1 \xleftarrow{\text{homog}}$$

$$z_{1H} = C_1 e^{2x}$$

$$(Axe^{2x})' = 2(Axe^{2x}) - C_2 e^{2x}$$

$$Ae^{2x} + 2Axe^{2x} = 2Axe^{2x} - C_2 e^{2x}$$

$$\Rightarrow A = -C_2$$

$$z_1 = z_{1H} + z_{1p}$$

$$= C_1 e^{2x} - C_2 x e^{2x}$$

So $\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} C_1 e^{2x} - C_2 x e^{2x} \\ C_2 e^{2x} \end{pmatrix}$

thus a f.s.s. for $\vec{z}' = [T]_{\mathcal{W}}^{\mathcal{W}} \vec{z}$ is $\left\{ \begin{pmatrix} e^{2x} \\ 0 \end{pmatrix}, \begin{pmatrix} -xe^{2x} \\ e^{2x} \end{pmatrix} \right\}$

and then a f.s.s. for $\vec{y}' = [T]_{\mathcal{S}}^{\mathcal{S}} \vec{y} = A\vec{y}$ is found by $\vec{y} = [I]_{\mathcal{W}}^{\mathcal{S}} \vec{z}$, giving

$$\left\{ \begin{pmatrix} e^{2x} \\ 3e^{2x} \end{pmatrix}, \begin{pmatrix} -xe^{2x} \\ -3xe^{2x} + e^{2x} \end{pmatrix} \right\}$$

5. *(20 pts)* Find a pair of matrix equations whose solution (if possible) would result in the vectors $\vec{a}$ and $\vec{b}$ that make $\vec{y}_p = (\vec{a} + \vec{b}x)e^{rx}$ a particular solution to the system

$$\vec{y}' = A\vec{y} + xe^{rx}\vec{v}$$

What is the condition on r that would guarantee that the solution would be possible?

Putting this form into the equation, we get

$$\left((\vec{a}+\vec{b}x)e^{rx}\right)' = A\left((\vec{a}+\vec{b}x)e^{rx}\right) + xe^{rx}\vec{v}$$

$$\vec{b}e^{rx} + (\vec{a}+\vec{b}x)re^{rx} = A\vec{a}e^{rx} + A\vec{b}xe^{rx} + \vec{v}xe^{rx}$$

$$\underbrace{\left(\vec{b} + r\vec{a} - A\vec{a}\right)}_{\boxed{(A-rI)\vec{a} = \vec{b}}} + x\underbrace{\left(r\vec{b} - A\vec{b} - \vec{v}\right)}_{\boxed{(A-rI)\vec{b} = -\vec{v}}} = \vec{0}$$

These equations can definitely be solved for $\vec{a}$ and $\vec{b}$ if $(A-rI)$ is nonsingular, a.k.a. if r is not an eigenvalue of A.