EXAM 2

Math 216, 2015-2016 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA____________ Day/Time____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________

1. ________
2. ________
3. ________
4. ________
5. ________

Total Score ________ (/100 points)

1. *(20 pts)* Your friend Bob is wondering if the functions $f(x) = \sin(x)$, $g(x) = \sin(x - \frac{\pi}{4})$, and $h(x) = \sin(x - \frac{\pi}{3})$ are linearly independent.

(a) Use a dimension argument to give Bob a definitive answer to his question. *(Be sure to be thorough with your explanation.)*

Noting that $\sin(x-\phi) = \sin x \cos\phi - \cos x \sin\phi$, we conclude that all three of these functions are in $\text{span}\{\sin x, \cos x\}$, which is two dimensional. Since $3>2$, the vectors are dependent.

(b) Use the Wronskian to draw the same conclusion. *(Be sure to be thorough with your explanation.)*

$$W(x) = \det\begin{pmatrix} \sin x & \sin(x-\pi/4) & \sin(x-\pi/3) \\ \cos x & \cos(x-\pi/4) & \cos(x-\pi/3) \\ -\sin x & -\sin(x-\pi/4) & -\sin(x-\pi/3) \end{pmatrix}$$

This is zero for all x, as the first and third rows are multiples of each other.

And f,g,h are all analytic, so we conclude that f,g,h are dependent.

2. *(20 pts)*

(a) Find all of the roots of the polynomial $f(x) = x^3 - x^2 + 8x + 10$.

Possible rat'l roots are $\pm 1, \pm 2, \pm 5, \pm 10$; $p(-1) = 0$, so $(x+1)$ is a factor.

$$\begin{array}{r l} & x^2 - 2x + 10 \\ x+1 \,\big)\!\! & \overline{x^3 - x^2 + 8x + 10} \\ & \underline{x^3 + x^2} \\ & -2x^2 + 8x + 10 \\ & \underline{-2x^2 - 2x} \\ & 10x + 10 \\ & \underline{10x + 10} \\ & 0 \end{array}$$

By the quadratic equation, we also have roots

$$x = \frac{2 \pm \sqrt{4 - 40}}{2} = 1 \pm 3i$$

So $f(x) = (x+1)\left(x - (1+3i)\right)\left(x - (1-3i)\right)$

(b) The differential equation $L(y) = 0$ has characteristic polynomial below. Find a fundamental set of real solutions to this equation.

$$p(\lambda) = (\lambda - 3)(\lambda - 2)^3 (\lambda - (4 - 5i))^2 (\lambda - (4 + 5i))^2$$

By theorems from class, a f.s.s. is

$$\left\{ e^{3x}, e^{2x}, xe^{2x}, x^2 e^{2x}, e^{4x}\cos 5x, e^{4x}\sin 5x, xe^{4x}\cos 5x, xe^{4x}\sin 5x \right\}$$

3. *(20 pts)* Find the form (but do not evaluate the coefficients) of a particular solution to the differential equation below, for which the characteristic polynomial is $p(\lambda) = (\lambda+1)(\lambda^2-4\lambda+5)$.

$$= (\lambda+1)(\lambda-(2+i)) \cdot (\lambda-(2-i))$$

$$L(y) = \underbrace{x}_{g_1} - \underbrace{3e^{2x}\cos(x)}_{g_2}$$

For $L(y_{p_1}) = g_1 = x$, we have $r_1 = 0$, which is <u>not</u> a root of $p(\lambda)$.

So $y_{p_1} = Ax + B$.

For $L(y_{p_2}) = g_2 = -3e^{2x}\cos x$, we have $r_2 = 2+i$, which <u>is</u> a root of $p(\lambda)$, $m_2 = 1$.

So $y_{p_2} = Cxe^{2x}\cos x + Dxe^{2x}\sin x$.

Then $y_p = y_{p_1} + y_{p_2}$

$= Ax + B + Cxe^{2x}\cos x + Dxe^{2x}\sin x$

4. *(20 pts)* Some of the solutions to the differential equation below are products of exponentials and sine waves. What are the two possible frequencies (f, not ω!) for these sine waves?

$$y^{[5]} + 32y = 0$$

$p(\lambda) = \lambda^5 + 32$, so we are looking for the 5th roots of -32. Clearly -2 is one, so the others are

$$\lambda = -2,\ -2e^{2\pi i/5},\ -2e^{4\pi i/5},\ -2e^{6\pi i/5},\ -2e^{8\pi i/5}$$

$$= -2,\ 2e^{7\pi i/5},\ 2e^{9\pi i/5},\ 2e^{1\pi i/5},\ 2e^{3\pi i/5}$$

$$= -2,\ 2e^{\pm 1\pi i/5},\ 2e^{\pm 3\pi i/5}$$

$$= -2,\ \left(2\cos\tfrac{\pi}{5}\right) \pm i\left(2\sin\tfrac{\pi}{5}\right),\ \left(2\cos\tfrac{3\pi}{5}\right) \pm i\left(2\sin\tfrac{3\pi}{5}\right)$$

Then $\omega_1 = 2\sin\frac{\pi}{5}$, $\omega_2 = 2\sin\frac{3\pi}{5}$

And $f = \frac{\omega}{2\pi}$, so

$$f_1 = \frac{1}{\pi}\sin\frac{\pi}{5},\quad f_2 = \frac{1}{\pi}\sin\frac{3\pi}{5}$$

5. *(20 pts)* As was discussed in class, let $S = \{\text{solutions to } L(y) = 0\}$, and let $T : S \to \mathbb{R}^n$ be the one-to-one and onto function defined by

$$T(y) = \begin{pmatrix} y(0) \\ y'(0) \\ \vdots \\ y^{[n-1]}(0) \end{pmatrix}$$

Show that T is a linear transformation.

$$T(c_1 y_1 + c_2 y_2)$$

$$= \begin{pmatrix} (c_1 y_1 + c_2 y_2)(0) \\ (c_1 y_1 + c_2 y_2)'(0) \\ \vdots \\ (c_1 y_1 + c_2 y_2)^{[n-1]}(0) \end{pmatrix}$$

$$= \begin{pmatrix} (c_1 y_1 + c_2 y_2)(0) \\ (c_1 y_1' + c_2 y_2')(0) \\ \vdots \\ (c_1 y_1^{[n-1]} + c_2 y_2^{[n-1]})(0) \end{pmatrix}$$

$$= \begin{pmatrix} c_1 y_1(0) + c_2 y_2(0) \\ c_1 y_1'(0) + c_2 y_2'(0) \\ \vdots \\ c_1 y_1^{[n-1]}(0) + c_2 y_2^{[n-1]}(0) \end{pmatrix}$$

$$= c_1 \begin{pmatrix} y_1(0) \\ y_1'(0) \\ \vdots \\ y_1^{[n-1]}(0) \end{pmatrix} + c_2 \begin{pmatrix} y_2(0) \\ y_2'(0) \\ \vdots \\ y_2^{[n-1]}(0) \end{pmatrix} = c_1 T(y_1) + c_2 T(y_2)$$