EXAM 1

Math 216, 2015-2016 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________________ Day/Time________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. ________

2. ________

3. ________

4. ________

5. ________

Total Score ________ (/100 points)

1. *(20 pts)* The system $(A|I)$ is row equivalent to

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{R}\!\!\left|\underbrace{\begin{matrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{matrix}}_{E}\right), \quad \text{and note that} \quad \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -4 & 2 \\ 0 & 3 & -1 \\ 1 & 2 & -1 \end{pmatrix}$$

(a) Find the complete set of solutions to the system $A\vec{x} = \vec{b}_1$, where $\vec{b}_1 = (-4, 3, 2)$.

$A\vec{x} = \vec{b}$ reduces to $R\vec{x} = E\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 3 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \begin{array}{l} x_1 = -2x_3 - x_4 \\ x_2 = 1 - 3x_3 - 5x_4 \end{array}$$

Then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_3 - x_4 \\ 1 - 3x_3 - 5x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

(b) Your friend Bob says that in general the system $A\vec{x} = \vec{b}$ has solutions if and only if $\vec{b}$ is a linear combination of $(-1, 0, 1)$ and $(-4, 3, 2)$. Is he right? Explain why or why not.

$A\vec{x} = \vec{b}$ has solutions $\iff$ $R\vec{x} = E\vec{b}$ has solutions

$\iff$ $E\vec{b}$ has third component $= 0$

$\iff$ $E\vec{b} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$

$\iff$ $\vec{b} = E^{-1}\begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$

$\iff$ $\vec{b} = c_1\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

$\iff$ $\vec{b}$ is a l.c. of $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix}$.

2. *(20 pts)* Use a row reduction to compute the inverse of the matrix A below, and to find elementary matrices E_1, E_2, E_3 such that $A = E_3 E_2 E_1$.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\Big\downarrow \; M_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) & \begin{array}{l} (3) \\ (2) \\ (1) \end{array} \end{array}$$

$$\Big\downarrow \; M_2 = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 0 & 1 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) & \begin{array}{l} (1) - 3(3) \\ (2) \\ (3) \end{array} \end{array}$$

$$\Big\downarrow \; M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) & \begin{array}{l} (1) \\ (2)/4 \\ (3) \end{array} \end{array}$$

This is I, so this is A^{-1}.

The row reduction shows $M_3 M_2 M_1 A = I$, so $A = M_1^{-1} M_2^{-1} M_3^{-1}$; so we can choose

$$E_1 = M_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_2 = M_2^{-1} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_3 = M_1^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

3. *(16 pts)* The three vectors below are the edge vectors of the parallelepiped P.

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

Q is the parallelepiped whose edge vectors are $A\vec{v}_1$, $A\vec{v}_2$, $A\vec{v}_3$, where A is given by

$$A = \begin{pmatrix} 3 & 2 & 3 \\ 0 & 0 & 2 \\ 3 & 5 & 4 \end{pmatrix}$$

Compute the volume of Q <u>*without*</u> computing its edge vectors.

Volume of P is $\left| \det \begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 2 & 5 \end{pmatrix} \right| = |3(18) - 1(3)| = 51$

$$(\text{Volume of } Q) = |\det A| \, (\text{Volume of } P)$$

$$= |(-2)(9)| \, (51) = 918$$

<u>Alternate solution:</u>

$$(\text{Volume of } Q) = \left| \det \begin{pmatrix} | & | & | \\ A\vec{v}_1 & A\vec{v}_2 & A\vec{v}_3 \\ | & | & | \end{pmatrix} \right|$$

$$= \left| \det \left((A) \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix} \right) \right|$$

$$= \left| (\det A) \left(\det \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix} \right) \right|$$

$$= |(-2)(9)| \, (51) = 918$$

4. *(24 pts)*

(a) Compute the determinant of the matrix below with as little arithmetic as possible.

$$\begin{pmatrix} 4 & 3 & 5 & 0 & 2 \\ 1 & 6 & 12 & 3 & 11 \\ 0 & 2 & 0 & 0 & 0 \\ 4 & 2 & 5 & 0 & 2 \\ 5 & 1 & 4 & 0 & 2 \end{pmatrix}$$

Cofactor expansion along 4th column gives

$$\det = 3\det\begin{pmatrix} 4 & 3 & 5 & 2 \\ 0 & 2 & 0 & 0 \\ 4 & 2 & 5 & 2 \\ 5 & 1 & 4 & 2 \end{pmatrix}$$; then along 2nd row gives

$$= 3 \cdot 2 \cdot \det\begin{pmatrix} 4 & 5 & 2 \\ 4 & 5 & 2 \\ 5 & 4 & 2 \end{pmatrix}$$ ← two identical rows

$$= 3 \cdot 2 \cdot 0$$

$$= 0$$

(b) The matrix M below has determinant equal to 2. Compute the number in the 3rd row and 4th column of M^{-1}.

$$M = \begin{pmatrix} 2 & 1 & 31 & 5 \\ 0 & 1 & 12 & 3 \\ 1 & 1 & 10 & 8 \\ 23 & 15 & 565 & 10 \end{pmatrix}$$

$$M^{-1} = \frac{\operatorname{adj} M}{\det M} = \frac{\operatorname{adj} M}{2}$$

$$(M^{-1})_{34} = \frac{1}{2}(\operatorname{adj} M)_{34} = \frac{1}{2}(C_{43})$$

$$= \frac{(-1)^{4+3}}{2}\det\begin{pmatrix} 2 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 1 & 8 \end{pmatrix} = \frac{-1}{2}\big((1)(11) - (3)(1)\big)$$

$$= -4$$

5. *(20 pts)* Prove that if a list of vectors $\vec{v}_1, \ldots, \vec{v}_4 \in \mathbb{R}^4$ is linearly dependent, then that list cannot span $\mathbb{R}^4$.

$\{\vec{v}_1, \ldots, \vec{v}_4\}$ is l.d. $\iff c_1\vec{v}_1 + \ldots + c_4\vec{v}_4 = \vec{0}$ $\quad c_i$ not all zero

$$\iff \underbrace{\begin{pmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ \vdots \\ c_4 \end{pmatrix} = \vec{0} \qquad c_i \text{ not all zero}$$

$\iff$ A does not have the uniqueness property

$\iff$ rref(A) has ≤ 3 pivots

$\iff$ A does not have existence property

$\iff$ there is a $\vec{b}$ where $A\vec{x} = \vec{b}$ has no solutions

$\iff$ there is a $\vec{b}$ for which

$x_1\vec{v}_1 + \ldots + x_4\vec{v}_4 = \vec{b}$

has no solutions

$\iff$ the list does not span $\mathbb{R}^4$