EXAM 3

Math 216, 2015-2016 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________ Day/Time________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________

1. ________
2. ________
3. ________
4. ________
5. ________
6. ________

Total Score ________ (/100 points)

1. *(20 pts)* The matrix

$$A = \begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix}$$

has eigenvalues 3 and 4. Find an invertible matrix B and diagonal matrix D with $D = B^{-1}AB$.

$$(A - 3I) = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}$$ ← rank = 1 & $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in NS$,

so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the only e-vect.

$$(A - 4I) = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix}$$ ← rank = 1 & $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \in NS$,

so $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is the only e-vect.

So $$\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = A\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

and then

$$\underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}}_{D} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^{-1} A \underbrace{\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}}_{B}$$

2. *(20 pts)* In the inner product space $C[0,1]$ using the L^2 inner product, compute the cosine of the angle between the vectors $f(x) = x$ and $g(x) = x^2$.

$$\|f\|^2 = \int_0^1 (x)(x)\,dx = \int_0^1 x^2\,dx = 1/3$$

$$\|g\|^2 = \int_0^1 (x^2)(x^2)\,dx = \int_0^1 x^4\,dx = 1/5$$

$$\langle f,g\rangle = \int_0^1 (x)(x^2)\,dx = \int_0^1 x^3\,dx = 1/4$$

$$\cos\theta = \frac{\langle f,g\rangle}{\|f\|\,\|g\|} = \frac{1/4}{1/\sqrt{3}\cdot 1/\sqrt{5}} = \boxed{\frac{\sqrt{15}}{4}}$$

3. *(15 pts)* The function $L : V \to \mathbb{R}^3$ is a linear transformation.

(a) Use L to form a "Wronskian" to help us consider the potential linear independence / dependence of a list of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ in the vector space V.

$$W(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}) = \det \begin{pmatrix} | & | & | \\ L(\vec{v}_1) & L(\vec{v}_2) & L(\vec{v}_3) \\ | & | & | \end{pmatrix}$$

(b) Suppose we know the following:

$$L(\vec{v}_1) = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad L(\vec{v}_2) = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \quad \text{and} \quad L(\vec{v}_3) = \begin{pmatrix} 0 \\ 1 \\ 9 \end{pmatrix}$$

What can you conclude about the potential linear independence/dependence of the list of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

$$W = \det \begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & 5 & 9 \end{pmatrix} = -18$$

This is nonzero, so the list is independent.

(c) Suppose we know the following:

$$L(\vec{v}_4) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad L(\vec{v}_5) = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} \quad \text{and} \quad L(\vec{v}_6) = \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix}$$

What can you conclude about the potential linear independence/dependence of the list of vectors $\{\vec{v}_4, \vec{v}_5, \vec{v}_6\}$?

$$W = \det \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ 1 & 5 & -4 \end{pmatrix} = 0$$

This is zero, so we can conclude nothing.

4. *(15 pts)* Suppose we know that $A = B^{-1}CB$, and $\vec{q}\,'(x) = C\vec{q}(x)$. Find (and justify) a solution to the equation $\vec{y}\,'(x) = A\vec{y}(x)$.

$$\vec{y}\,' = B^{-1}CB\vec{y}$$

$$\underbrace{(B\vec{y})'}_{\vec{q}} = C\underbrace{(B\vec{y})}_{\vec{q}}$$

So $\vec{q} = B\vec{y}$, and thus

$\vec{y} = B^{-1}\vec{q}$ is a solution.

5. *(15 pts)* The arithmetic below is given.

$$A = \begin{pmatrix} -11 & -18 \\ 12.5 & 19 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2.5 & -1.5 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$$

Find a fundamental set of solutions to the system $\vec{y}\,' = A\vec{y}$.

$$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{V}}^{\mathcal{S}}\,[T]_{\mathcal{V}}^{\mathcal{V}}\,[I]_{\mathcal{S}}^{\mathcal{V}}$$

$$A \;=\; P \quad J \quad P^{-1}$$

J is in Jordan form, and $\left\{ \underbrace{\begin{pmatrix} -3 \\ 2.5 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 2 \\ -1.5 \end{pmatrix}}_{\vec{v}_2} \right\}$ is the Jordan basis.

$$e^{xA}\vec{v}_1 = e^{\lambda x}\vec{v}_1 = e^{4x}\begin{pmatrix} -3 \\ 2.5 \end{pmatrix}$$

$$e^{xA}\vec{v}_2 = e^{\lambda x}(\vec{v}_2 + x\vec{v}_1) = e^{4x}\begin{pmatrix} 2-3x \\ -1.5+2.5x \end{pmatrix}$$

This is a fundamental set of solutions.

6. *(15 pts)* Find a particular solution to the system below.

$$\begin{aligned} y_1' &= 3y_1 - 2y_2 + 1 \\ y_2' &= 2y_1 + y_2 + e^x \end{aligned}$$

$\vec{y}' = A\vec{y} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ e^x \end{pmatrix}$

Guess: $\vec{y}_p = \vec{a} + e^x \vec{b}$

$e^x \vec{b} = A(\vec{a} + e^x \vec{b}) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^x \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{b} = A\vec{b} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\quad 0 = A\vec{a} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$A\vec{a} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$(A - I)\vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 2 & -2 \\ 2 & 0 \end{pmatrix} \vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $\quad \begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix} \vec{a} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\vec{b} = \dfrac{\begin{pmatrix} 0 & 2 \\ -2 & 2 \end{pmatrix}}{4} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $\quad \vec{a} = \dfrac{\begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}}{7} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\vec{b} = \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}$ $\quad \vec{a} = \begin{pmatrix} -1/7 \\ 2/7 \end{pmatrix}$

So $\vec{y}_p = \begin{pmatrix} -1/7 \\ 2/7 \end{pmatrix} + e^x \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}$