EXAM 2

Math 216, 2015-2016 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________________ Day/Time________________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ________________

1. ________

2. ________

3. ________

4. ________

5. ________

Total Score ________ (/100 points)

1. *(20 pts)* Bob is interested in finding a fundamental set of solutions to the linear differential equation $L(y) = 0$. The equation is third order, and satisfies the conditions of the existence/uniqueness theorem. By trial and error Bob has found solutions $p(x)$, $q(x)$, $r(x)$, and $s(x)$, and he has computed that

$$A = \begin{pmatrix} p(0) & q(0) & r(0) & s(0) \\ p'(0) & q'(0) & r'(0) & s'(0) \\ p''(0) & q''(0) & r''(0) & s''(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 0 & 2 \\ 1 & 3 & 5 & 0 \end{pmatrix}$$

(a) What is the dimension of the set of solutions to this differential equation? Explain what specifically about the given information allows you to draw this conclusion.

The equation is third order, and linear and satisfies the conditions of the existence/uniqueness theorem, so the dimension is 3.

(b) Bob is considering the possibility that $\{p, q, s\}$ might be a fundamental set of solutions. Compute the value of the Wronskian, at $x = 0$, for this list of functions. Is this a fundamental set of solutions?

$$W(0) = \det\begin{pmatrix} p(0) & q(0) & s(0) \\ p'(0) & q'(0) & s'(0) \\ p''(0) & q''(0) & s''(0) \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 3 & 0 \end{pmatrix} = 11$$

This tells us $\{p,q,s\}$ is linearly independent in a 3-dim. vector space, so it is a fundamental set of solutions.

(c) What are the initial values (at $x = 0$) of the function $f(x) = p(x) - 2q(x) + r(x)$? What are the inital values of the function $z(x) = 0$? What then does the existence/uniqueness theorem then tell us about f, and what does this tell us about the question of whether $\{p, q, r\}$ might be a fundamental set of solutions?

$$\begin{pmatrix} f(0) \\ f'(0) \\ f''(0) \end{pmatrix} = \begin{pmatrix} p(0) \\ p'(0) \\ p''(0) \end{pmatrix} - 2\begin{pmatrix} q(0) \\ q'(0) \\ q''(0) \end{pmatrix} + \begin{pmatrix} r(0) \\ r'(0) \\ r''(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} z(0) \\ z'(0) \\ z''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ also. This initial condition has exactly one solution, so $f(x) = z(x) = 0$, and thus $p - 2q + r = 0$

So $\{p,q,r\}$ is linearly dependent, and thus not a fundamental set of solutions.

2. *(20 pts)* Find a fundamental set of solutions to the differential equation

$$y'''(x) + 5y''(x) + 4y'(x) - 10y(x) = 0$$

$p(\lambda) = \lambda^3 + 5\lambda^2 + 4\lambda - 10$ ← poss. rat. roots : $\pm 1, \pm 2, \pm 5, \pm 10$

$p(1) = 0$, so $(\lambda - 1)$ is a factor

$$\begin{array}{r|l} & \lambda^2 + 6\lambda + 10 \\ \lambda - 1 & \lambda^3 + 5\lambda^2 + 4\lambda - 10 \\ & \underline{\lambda^3 - \lambda^2} \\ & 6\lambda^2 + 4\lambda - 10 \\ & \underline{6\lambda^2 - 6\lambda} \\ & 10\lambda - 10 \\ & \underline{10\lambda - 10} \\ & 0 \end{array}$$

$q(x)$, has roots

$$\frac{-6 \pm \sqrt{36 - 40}}{2} = -3 \pm i$$

So $p(\lambda)$ has roots : 1, $-3+i$, $-3-i$

Then a fundamental set of solutions is

$$\left\{ e^{x} , e^{-3x}\cos x , e^{-3x}\sin x \right\}$$

3. *(20 pts)* Find the form (but do NOT evaluate the coefficients!) of a particular solution to the differential equation

$$y'''(x) + 9y'(x) = x^2 - x\cos 3x$$

$p(\lambda) = \lambda^3 + 9\lambda = (\lambda)(\lambda + 3i)(\lambda - 3i)$

has roots: $0, -3i, 3i$.

For $g_1 = x^2$, we have $r_1 = 0$, is a root of p, with $m_1 = 1$

So we guess $y_{p_1} = x(c_2x^2 + c_1x + c_0)$

For $g_2 = -x\cos 3x$, we have $r_2 = 3i$, is a root of p, with $m_2 = 1$

So we guess $y_{p_2} = x(d_1x + d_0)\cos 3x + x(e_1x + e_0)\sin 3x$

Then by linearity, our combined guess is

$$\begin{aligned} y_p &= y_{p_1} + y_{p_2} \\ &= x(c_2x^2 + c_1x + c_0) \\ &\quad + x(d_1x + d_0)\cos 3x \\ &\quad + x(e_1x + e_0)\sin 3x \end{aligned}$$

4. *(20 pts)* The general solution to the differential equation describing the motion of a particular mass on a particular spring is

$$y(t) = c_1 \cos 3t + c_2 \sin 3t + 5 \sin 2t$$

(a) Find the solution y_1 that satisfies the initial condition $y_1(0) = 1$, $y_1'(0) = 7$.

$$y'(t) = -3c_1 \sin 3t + 3c_2 \cos 3t + 10 \cos 2t$$

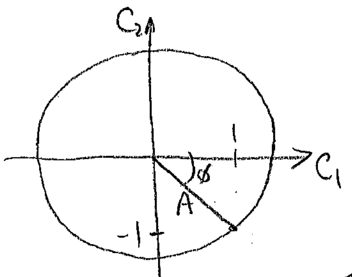
So

$$\left.\begin{aligned} y(0) &= c_1 = 1 \\ y'(0) &= 3c_2 + 10 = 7 \end{aligned}\right\} \Longrightarrow c_1 = 1,\ c_2 = -1$$

So $y_1 = \cos 3t - \sin 3t + 5 \sin 2t$

(b) Rewrite the above solution y_1 as the sum of two sinusoids of different frequency.

$$A \cos(3t - \phi) = A\cos\phi \cos 3t + A \sin\phi \sin 3t$$
$$= (1) \cos 3t + (-1) \sin 3t$$



So $A = \sqrt{2}$, $\phi = -\pi/4$.

Then $y_1 = \sqrt{2} \cos(3t + \pi/4) + 5 \sin 2t$

(c) Rewrite the (different) solution $y_2 = 5 \sin 3t + 5 \sin 2t$ as a product of two sinusoids.

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$\underbrace{\sin(a+b)}_{3t} + \underbrace{\sin(a-b)}_{2t} = 2 \sin a \cos b$$

$$\Longrightarrow a = \frac{5t}{2},\ b = \frac{t}{2}$$

$$\sin 3t + \sin 2t = 2 \sin\left(\frac{5t}{2}\right) \cos\left(\frac{t}{2}\right)$$

So

$$5 \sin 3t + 5 \sin 2t = 10 \sin\left(\frac{5t}{2}\right) \cos\left(\frac{t}{2}\right)$$

5. *(20 pts)* In this question we consider the vector space V with basis $\mathcal{V} = \{\sin x, \cos x\}$, and the alternative basis $\mathcal{W} = \{\sin x - \cos x, \sin x + \cos x\}$.

(a) Find the change of basis matrices $[I]_{\mathcal{V}}^{\mathcal{W}}$ and $[I]_{\mathcal{W}}^{\mathcal{V}}$.

$$\sin x - \cos x = (1)\sin x + (-1)\cos x \Rightarrow [\sin x - \cos x]_{\mathcal{V}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\sin x + \cos x = (1)\sin x + (1)\cos x \Rightarrow [\sin x + \cos x]_{\mathcal{V}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So $[I]_{\mathcal{W}}^{\mathcal{V}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Then $[I]_{\mathcal{V}}^{\mathcal{W}} = \left([I]_{\mathcal{W}}^{\mathcal{V}}\right)^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$

(b) Compute $[D^2 - 3]_{\mathcal{V}}^{\mathcal{V}}$.

$$(D^2-3)(\sin x) = -\sin x - 3\sin x = -4\sin x$$

$$\Rightarrow [(D^2-3)(\sin x)]_{\mathcal{V}} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$$

$$(D^2-3)(\cos x) = -\cos x - 3\cos x = -4\cos x$$

$$\Rightarrow [(D^2-3)(\cos x)]_{\mathcal{V}} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

Then $[D^2-3]_{\mathcal{V}}^{\mathcal{V}} = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$

(c) Use the previous parts of this question to compute $[D^2 - 3]_{\mathcal{W}}^{\mathcal{W}}$.

$$[D^2-3]_{\mathcal{W}}^{\mathcal{W}} = [I]_{\mathcal{V}}^{\mathcal{W}} [D^2-3]_{\mathcal{V}}^{\mathcal{V}} [I]_{\mathcal{W}}^{\mathcal{V}}$$

$$= \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$$