EXAM 1

Math 216, 2015-2016 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number__________ TA____________________ Day/Time____________________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ____________________

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________

Total Score __________ (/100 points)

1. *(20 pts)* The augmented matrix $(A|\vec{b}_1)$ row reduces to

$$\left(\begin{array}{ccccc|c} 0 & 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longleftarrow \text{no contradictions!}$$

(a) Find the complete set of solutions to the system $A\vec{x} = \vec{b}_1$.

$$x_2 = 1 - 3x_3 - 2x_5 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ 1 - 3x_3 - 2x_5 \\ x_3 \\ 3 - 5x_5 \\ x_5 \end{pmatrix}$$

$$x_4 = 3 - 5x_5$$

(b) Your friend Bob says that, for a vector $\vec{b}_2$ he has found, the system $A\vec{x} = \vec{b}_2$ has a unique solution. Is there enough information available to decide if Bob is right or wrong? Explain.

Bob is wrong. There are free variables in the system, so $A\vec{x} = \vec{b}_2$ must have either no solutions or infinitely many solutions.

(c) The system of equations $A\vec{x} = \vec{b}_3$ is satisfied by $\vec{s} = (2, 1, 5, 3, 1)$. Find the complete set of solutions to the system.

$A\vec{x} = \vec{0}$ reduces to $\left(\begin{array}{ccccc|c} 0 & 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$, so the homogeneous solutions are $\begin{pmatrix} x_1 \\ -3x_3 - 2x_5 \\ x_3 \\ -5x_5 \\ x_5 \end{pmatrix}$.

Particular + homogeneous then gives $\vec{x} = \begin{pmatrix} 2 + x_1 \\ 1 - 3x_3 - 2x_5 \\ 5 + x_3 \\ 3 - 5x_5 \\ 1 + x_5 \end{pmatrix}$.

2. *(20 pts)* The systems

$$\left(\begin{array}{c|c} A & \begin{matrix}1\\0\\0\end{matrix}\end{array}\right) \quad \left(\begin{array}{c|c} A & \begin{matrix}0\\1\\0\end{matrix}\end{array}\right) \quad \left(\begin{array}{c|c} A & \begin{matrix}0\\0\\1\end{matrix}\end{array}\right)$$

row reduce to the systems below, respectively:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 3\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 5\end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1\\ 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 0\end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 2 & 0 & 2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 4\end{array}\right)$$

(a) Find a row reduction matrix E for which EA is the reduced row echelon form in the systems above.

If row reduction is left multiplication by $E = E_k \cdots E_1$,

then $E\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}3\\2\\5\end{pmatrix}$, $E\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\3\\0\end{pmatrix}$, $E\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\4\end{pmatrix}$.

So $E = \begin{pmatrix}3 & 1 & 2\\ 2 & 3 & 1\\ 5 & 0 & 4\end{pmatrix}$

(b) Find the complete set of solutions to the system $A\vec{x} = \vec{b}$, with $\vec{b} = (1, 2, 3)$.

Row reducing with E, $A\vec{x} = \vec{b}$ becomes $R\vec{x} = E\vec{b}$, or

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 11\\ 0 & 0 & 1 & 11\\ 0 & 0 & 0 & 17\end{array}\right)$$

There is a contradiction, so there are no solutions.

(c) Find another matrix $E_2 \neq E$ for which $E_2 A$ is the same reduced row echelon form in the systems above. *(Hint: Think about one additional row operation, that would not change the above reduced row echelon form.)*

$$\underbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}}_{F} \underbrace{\begin{pmatrix}1 & 2 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}}_{R = EA} = \underbrace{\begin{pmatrix}1 & 2 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}}_{R = EA}$$

So $F(EA) = R$, and then $(FE)A = R$.

Then $E_2 = FE = \begin{pmatrix}3 & 1 & 2\\ 2 & 3 & 1\\ 10 & 0 & 8\end{pmatrix}$ reduces A to R.

3. *(10 pts)* Use Cramer's rule to find an algebraic expression for R in terms of the other variables in this system (it is known that $ac \neq 1$).

$$\begin{pmatrix} 1 & a & 0 \\ 0 & b & 1 \\ c & 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ R \\ K \end{pmatrix} = \begin{pmatrix} L \\ P \\ m \end{pmatrix}$$

$$R = \frac{\det\begin{pmatrix} 1 & L & 0 \\ 0 & P & 1 \\ c & M & 0 \end{pmatrix}}{\det\begin{pmatrix} 1 & a & 0 \\ 0 & b & 1 \\ c & 1 & 0 \end{pmatrix}} = \frac{Lc - M}{ac - 1}$$

cofactor expansions along third columns.

4. *(10 pts)* The 3×3 matrix A has rows $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, as indicated below.

$$\begin{pmatrix} - & \vec{r}_1 & - \\ - & \vec{r}_2 & - \\ - & \vec{r}_3 & - \end{pmatrix}$$

The matrix B is obtained from A by the row operation that adds 5 times the third row to the second row.

Use multilinearity explicitly to demonstrate that $\det(B) = \det(A)$.

$$\underbrace{\det\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 + 5\vec{r}_3 \\ \vec{r}_3 \end{pmatrix}}_{= \det B} = 1 \cdot \underbrace{\det\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{pmatrix}}_{= \det A} + 5 \cdot \underbrace{\det\begin{pmatrix} \vec{r}_1 \\ \vec{r}_3 \\ \vec{r}_3 \end{pmatrix}}_{= 0 \text{ b/c two identical rows}}$$

5. *(10 pts)* Is the list of polynomials below linearly dependent or linearly independent? Be sure to show all of the steps in your reasoning.

$$\{x^2 + x - 1, 2x + 1, 4x^2 - 3\}$$

$$c_1\left(x^2+x-1\right) + c_2\left(2x+1\right) + c_3\left(4x^2-3\right) = 0$$

$$\left(c_1+4c_3\right)x^2 + \left(c_1+2c_2\right)x + \left(-c_1+c_2-3c_3\right)\cdot 1 = 0$$

$$\begin{aligned} 1c_1 + 0c_2 + 4c_3 &= 0 \\ 1c_1 + 2c_2 + 0c_3 &= 0 \\ -1c_1 + 1c_2 - 3c_3 &= 0 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & 4 \\ 1 & 2 & 0 \\ -1 & 1 & -3 \end{pmatrix} \vec{c} = \vec{0}$$

det $= 6 \neq 0$, $\Rightarrow$ nonsingular

$\Rightarrow$ unique sols $\Rightarrow$ no signif rels

$\Rightarrow$ linearly independent.

6. *(10 pts)* Suppose that $\{\vec{p}, \vec{q}, \vec{r}\}$ is linearly dependent. Show that one of these vectors must be a linear combination of the other two.

$\{\vec{p}, \vec{q}, \vec{r}\}$ is l.d. $\iff$ $\exists\ c_1, c_2, c_3$ not all zero with $c_1\vec{p} + c_2\vec{q} + c_3\vec{r} = \vec{0}$

Since the coefficients are not all zero, at least one must be nonzero.

Suppose $c_1 \neq 0$. Then $\vec{p} = -\frac{c_2}{c_1}\vec{q} - \frac{c_3}{c_1}\vec{r}$, and one is a linear combination of the other two.

(Similarly for the other two cases, $c_2 \neq 0$ and $c_3 \neq 0$.)

7. *(20 pts)* Suppose S is the collection of all differentiable functions f for which $f'(0) = 3f'(1)$. Prove or disprove the assertion that S is a vector space.

$S \subset D'$, a known vector space.

To show S is a subspace of D' (and thus also a vector space), we check two conditions:

(1) Closed under addition:

We assume $g, h \in S$.

$$\Rightarrow \quad g'(0) = 3g'(1) \text{ and } h'(0) = 3h'(1)$$

$$\Rightarrow \quad g'(0) + h'(0) = 3\left(g'(1) + h'(1)\right)$$

$$\Rightarrow \quad (g+h)'(0) = 3(g+h)'(1) \Rightarrow g+h \in S.$$

(2) Closed under scalar multiplication:

We assume $g \in S$.

$$\Rightarrow \quad g'(0) = 3g'(1)$$

$$\Rightarrow \quad cg'(0) = 3cg'(1)$$

$$\Rightarrow \quad (cg)'(0) = 3(cg)'(1) \Rightarrow cg \in S$$

These two observations show S is a subspace, and thus a vector space.