EXAM 3

Math 216, 2014-2015 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA____________ Day/Time____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________

1. ________

2. ________

3. ________

4. ________

5. ________

Total Score ________ (/100 points)

1. *(20 pts)* Compute the inverse of the matrix A below without using a row reduction and without using determinants.

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Columns of A are orthogonal, have magnitude of $\sqrt{2}$, so $B = A/\sqrt{2}$ is an orthogonal matrix

$$B^{-1} = B^T = \left(A/\sqrt{2}\right)^T = A^T/\sqrt{2}$$

$$\parallel$$

$$\left(A/\sqrt{2}\right)^{-1} = \sqrt{2}\, A^{-1}$$

So $A^{-1} = A^T/2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \Big/ 2$

2. *(20 pts)* V is the inner product space consisting of vectors in $\mathbb{R}^3$, using the usual vector addition and scalar multilication, but with the inner product defined by

$$\langle \vec{v}, \vec{w} \rangle = (A\vec{v}) \cdot (A\vec{w}) = (A\vec{v})^T (A\vec{w}) \quad \text{where} \quad A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

Use the Gram-Schmidt method to find an orthonormal basis for the span of the vectors $(1, 0, 0)$ and $(0, 1, 0)$.

$$\left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\|^2 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 4$$

So $\left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\| = 2$, and we choose $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} / 2$

$$\left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix} \right\rangle = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 3$$

Then $\vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix}$ is $\perp$ to $\vec{v}_1$.

$$\|\vec{x}_2\|^2 = \left\langle \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix} \right\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1$$

So $\|\vec{x}_2\| = 1$, and we choose $\vec{v}_2 = \vec{x}_2 / 1 = \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix}$

Orthonormal basis is $\left\{ \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix} \right\}$

3. *(20 pts)* Your friend Bob is trying to find the matrix M. He knows of the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}$$

that $\vec{v}_1$ and $\vec{v}_3$ are eigenvectors of M with eigenvalue 3, and also that $M\vec{v}_2 = 3\vec{v}_2 + v_3$.

(a) Find the Jordan form F of M, and a basis that achieves this form (be careful to list the vectors in the order consistent with the Jordan form F).

$$M\vec{v}_3 = 3\vec{v}_3 + 0\vec{v}_2 + 0\vec{v}_1$$
$$M\vec{v}_2 = 1\vec{v}_3 + 3\vec{v}_2 + 0\vec{v}_1$$
$$M\vec{v}_1 = 0\vec{v}_3 + 0\vec{v}_2 + 3v_1$$

With $\mathcal{V} = \{\vec{v}_3, \vec{v}_2, \vec{v}_1\}$, we have Jordan form

$$F = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(b) Find matrices A and B, and choose between the expressions ABA^{-1} and $A^{-1}BA$, so that M is computed by the expression you chose.

$$M = [T]_{\mathcal{S}}^{\mathcal{S}}, \quad \vec{F} = [T]_{\mathcal{V}}^{\mathcal{V}}, \quad \text{and}$$

$$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{V}}^{\mathcal{S}} [T]_{\mathcal{V}}^{\mathcal{V}} [I]_{\mathcal{S}}^{\mathcal{V}}$$

so

$$M = [I]_{\mathcal{V}}^{\mathcal{S}} F \left([I]_{\mathcal{V}}^{\mathcal{S}}\right)^{-1}$$
$$= A \quad B \quad A^{-1}$$

where $B = F$, and $A = [I]_{\mathcal{V}}^{\mathcal{S}} = \begin{pmatrix} -1 & 2 & 1 \\ 5 & 0 & 3 \\ 0 & -2 & 2 \end{pmatrix}$

4. *(20 pts)* Find a fundamental set of solutions to the system

$$\begin{aligned} y_1' &= -7y_1 + 15y_2 \\ y_2' &= -6y_1 + 12y_2 \end{aligned}$$

$$A = \begin{pmatrix} -7 & 15 \\ -6 & 12 \end{pmatrix} \qquad \det(A - \lambda I) = (-7-\lambda)(12-\lambda) + 90$$

$$= \lambda^2 - 5\lambda + 6$$

$$= (\lambda - 2)(\lambda - 3)$$

$$\Rightarrow \lambda = 2 \text{ or } 3$$

$\lambda = 2$: $A - \lambda I = \begin{pmatrix} -9 & 15 \\ -6 & 10 \end{pmatrix}$ has NS $= \left\{ k \begin{pmatrix} 5 \\ 3 \end{pmatrix} \right\}$

so $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ is an eigenvector with eigenvalue 2.

$\lambda = 3$: $A - \lambda I = \begin{pmatrix} -10 & 15 \\ -6 & 9 \end{pmatrix}$ has NS $= \left\{ k \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$

so $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is an eigenvector with eigenvalue 3.

A is diagonalizable, and a f.s.s. is

$$\left\{ e^{2x} \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \ e^{3x} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$$

5. *(20 pts)* The matrices P and P^{-1} are given below.

$$P = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -3 & -1 \\ -2 & 4 & 1 \end{pmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \nwarrow \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}$$

We consider the system

$$\vec{y}' = \underbrace{\begin{pmatrix} -1 & 9 & 2 \\ -2 & 7 & 1 \\ 0 & 2 & 3 \end{pmatrix}}_{A} \vec{y}$$

(a) Use the basis formed by the columns of P to try to decouple the above system. Is the resulting system decoupled completely?

$$P^{-1}AP = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -3 & -1 \\ -2 & 4 & 1 \end{pmatrix} \begin{pmatrix} -1 & 9 & 2 \\ -2 & 7 & 1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 1 \\ 2 & -3 & -1 \\ -2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 10 & 9 \\ 0 & 3 & 4 \\ 6 & 8 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} = J$$

$$\vec{y}' = A\vec{y}$$

$$\vec{y}' = PJP^{-1}\vec{y}$$

$$(P^{-1}\vec{y})' = J\underbrace{(P^{-1}\vec{y})}_{\vec{z}}$$

Then $\vec{z}' = J\vec{z}$

This system is <u>not</u> decoupled because J is not diagonal.
(But J is in Jordan form!)

(b) Find the solution to the above equation with initial condition $\vec{y}(0) = (2, 1, 1)$ *(you may use any method from the course).*

$$\vec{y} = e^{xA}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = e^{xA}\vec{v}_3,$$

and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a Jordan basis.

Then $e^{xA}\vec{v}_3 = e^{3x}\left(\vec{v}_3 + x\vec{v}_2 + \frac{x^2}{2}\vec{v}_1\right)$

$$= e^{3x}\begin{pmatrix} 2 + 3x + \frac{1}{2}x^2 \\ 1 + 1x \\ 1 + 2x + 1x^2 \end{pmatrix}.$$