EXAM 2

Math 216, 2014-2015 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________________ Day/Time________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. ________

2. ________

3. ________

4. ________

5. ________

6. ________

Total Score ________ (/100 points)

1. *(15 pts)* The columns of the 5×4 matrix A are linearly independent, and B is an invertible 5×5 matrix. Show that the columns of BA are linearly independent. *(Hint: Consider the contrapositive.)*

To show the contrapositive, we suppose the columns of BA are dependent, so that

(nontrivial coefficients)

$$c_1 (BA)_1 + \ldots + c_4 (BA)_4 = \vec{0}$$

(columns of BA)

which we can rewrite as

$$BA\vec{c} = \vec{0}$$

Multiplying both sides by B^{-1}, we get

$$A\vec{c} = \vec{0}$$

which we can rewrite as

$$c_1 \vec{a}_1 + \ldots + c_4 \vec{a}_4 = \vec{0}$$

(columns of A)

which shows that the columns of A are dependent, as required.

2. *(20 pts)* Find a particular solution to the differential equation below. *(Hint: Consider a related complex equation.)*

$$y'' + 3y = x\cos x$$

The related complex equation is

$$z'' + 3z = xe^{ix}$$

We guess $\quad z = Axe^{ix} + Be^{ix}$

so $\quad z' = Ae^{ix} + iAxe^{ix} + iBe^{ix}$

and $\quad z'' = iAe^{ix} + iAe^{ix} - Axe^{ix} - Be^{ix}$

$$= (-A)xe^{ix} + (-B + 2iA)e^{ix}$$

and the equation becomes

$$((-A) + 3(A))xe^{ix} + ((-B+2iA) + 3(B))e^{ix} = xe^{ix}$$

so $\quad 2A = 1 \quad \Rightarrow \quad A = 1/2$

$$2B + 2iA = 0 \quad \Rightarrow \quad B = -i/2$$

and thus $\quad z = \left(\frac{1}{2}\right)xe^{ix} + \left(\frac{-i}{2}\right)e^{ix}$

$$= \frac{1}{2}x\cos x + \frac{1}{2}xi\sin x - \frac{1}{2}i\cos x + \frac{1}{2}\sin x$$

Then $y = \operatorname{Re}(z)$

$$= \boxed{\frac{1}{2}x\cos x + \frac{1}{2}\sin x}$$

3. *(15 pts)*

(a) The functions

$$\begin{aligned} y_1 &= 13e^{2x}\cos(3x) + 45e^{2x}\sin(3x) + 32e^{5x} \\ y_2 &= 25e^{2x}\cos(3x) + 31e^{2x}\sin(3x) + 64e^{5x} \\ y_3 &= 82e^{2x}\cos(3x) + 43e^{2x}\sin(3x) + 77e^{5x} \end{aligned}$$

form a linearly independent list, and each is a solution to the real constant coefficient differential equation below.

$$a_3y''' + a_2y'' + a_1y' + a_0y = 0$$

What fundamental set of real solutions to this differential equation is the *easiest* to justify given only the information above? Explain *completely* how you know this is in fact a fundamental set of solutions.

$\{y_1, y_2, y_3\}$ is an independent list of vectors in a 3 dimensional vector space of solutions (because the order of the LDE is 3), so this is a fundamental set.

(b) Find a fundamental set of solutions to the constant coefficient linear differential equation $L(y) = 0$ whose characteristic polynomial factors as

$$p(\lambda) = (\lambda^2 + 5)(\lambda + 4)^3(\lambda^2 + 2\lambda + 10)$$

roots are: $\pm\sqrt{5}i$, -4, $-1 \pm 3i$

multiplicity $=3$

Per theorems from class, a fundamental set of solutions is:

$$\{\cos\sqrt{5}x, \sin\sqrt{5}x, e^{-4x}, xe^{-4x}, x^2e^{-4x}, e^{-x}\cos 3x, e^{-x}\sin 3x\}$$

4. *(15 pts)* Your grandfather's old car has a suspension system of springs and "shock absorbers" (which provide a friction-like resistance), whose behavior can be related to a mass on a spring in a resistive medium in that it is described by the same sort of differential equation. For your grandfather's car, the modeling differential equation is

$$y''(t) + Ay'(t) + Ky(t) = 0$$

where A is proportional to the resistance of the shock absorbers and K is proportional to the strength of the springs.

The current shock absorbers on his car have $A = 2$. You have also measured that when the car hits a bump in the road the frequency ($f = \frac{\omega}{2\pi}$) of the resulting (decaying) oscillation of the car is $\frac{3}{2\pi}$.

Your grandfather wants to replace his shock absorbers with new ones such that A remains as low as possible (to keep the ride of the car soft and comfortable) while still completely eliminating the oscillation (using the same springs). What is the ideal value of A you should recommend for him to use?

With $A=2$, $y'' + 2y' + Ky = 0$

so $\lambda = -\frac{2}{2} \pm \frac{\sqrt{4-4K}}{2}$

Oscillation implies $4-4K < 0$, so this becomes

$$\lambda = -1 \pm i\sqrt{K-1}$$

and solutions are

$$y = Ae^{-t}\cos\left(\sqrt{K-1}\,t\right) + Be^{-t}\sin\left(\sqrt{K-1}\,t\right)$$

which have frequency

$$f = \frac{\sqrt{K-1}}{2\pi} = \frac{3}{2\pi}, \quad \text{so } K = 10$$

Using this $K=10$ and solving for a new A in

$$y'' + Ay' + 10y = 0$$

we want $A^2 - 40 = 0$ for critical damping,

so $\boxed{A = \sqrt{40} = 2\sqrt{10}.}$

5. *(15 pts)* Find a basis for the kernel of the linear transformation $(D^2 - 6D + 9) : C^\infty \to C^\infty$.

We want all solutions to

$$(D^2 - 6D + 9)y = 0$$

so $\quad y'' - 6y' + 9y = 0$

The characteristic polynomial is

$$p(\lambda) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

The root is 3 with multiplicity 2, so by theorems from class the fundamental set of solutions is

$$\{e^{3x}, xe^{3x}\}$$

This is a basis for the kernel.

6. *(20 pts)*

(a) What does it mean to say that the function $f : X \to Y$ is a linear transformation? (Be thorough!)

X and Y must both be vector spaces, and

$f(a\vec{x}_1 + b\vec{x}_2) = a f(\vec{x}_1) + b f(\vec{x}_2)$ for all $\vec{x}_1, \vec{x}_2 \in X$,

and $a, b \in \mathbb{R}$.

(b) Suppose that $P : A \to B$ and $Q : B \to C$ are linear transformations. Show that the product QP is a linear transformation.

$$
\begin{aligned}
QP(a\vec{x}_1 + b\vec{x}_2) \\
&= Q\left(P(a\vec{x}_1 + b\vec{x}_2)\right) \\
&= Q\left(aP(\vec{x}_1) + bP(\vec{x}_2)\right) && \leftarrow \text{by linearity of } P \\
&= aQ\left(P(\vec{x}_1)\right) + bQ\left(P(\vec{x}_2)\right) && \leftarrow \text{by linearity of } Q \\
&= a\,QP(\vec{x}_1) + b\,QP(\vec{x}_2)
\end{aligned}
$$