EXAM 1

Math 216, 2014-2015 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________________ Day/Time________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. ________
2. ________
3. ________
4. ________
5. ________
6. ________

Total Score ________ (/100 points)

1. *(20 pts)* Your friend Bob has worked on finding the solutions to the four systems of equations

$$A\vec{x} = \vec{b}_1 \quad \text{and} \quad A\vec{x} = \vec{b}_2 \quad \text{and} \quad A\vec{x} = \vec{b}_3 \quad \text{and} \quad A\vec{x} = \vec{b}_4$$

each of which has n equations and n unknowns with the same coefficient matrix A. Bob states that the first system has no solutions, the second system has a unique solution, the third system has exactly two solutions, and the fourth system has infinitely many solutions.

Given only the information above, what is the maximum possible number of his statements that could be correct? Which of the four statements would those be? Explain your reasoning fully.

Statement 3 is impossible.

Statements 1 and 4 are compatible, corresponding to the possibility that A is not nonsingular.

Statement 2 would require A to be nonsingular, so it is incompatible with 1 and 4.

So Bob could be right about at most $\boxed{\text{two}}$ of his statements, those being $\boxed{\text{1 and 4.}}$

2. *(15 pts)* The matrix A is defined as the product below.

$$A = \begin{pmatrix} 1 & 3 & 2 & 5 \\ 6 & 2 & 5 & 12 \\ 2 & 1 & 1 & 3 \\ 5 & 5 & 4 & 11 \end{pmatrix} \begin{pmatrix} 345 & 266 & 852 & 238 & 331 & 776 & 743 \\ 543 & 325 & 470 & 109 & 521 & 842 & 346 \\ 465 & 766 & 214 & 426 & 247 & 981 & 211 \\ 763 & 218 & 692 & 436 & 325 & 685 & 165 \end{pmatrix} \begin{array}{l} R_1 \\ \vdots \\ \\ \\ \end{array}$$

Referring to the rows of A with the notation A_i, show that $A_4 = A_1 + 2A_3$.

Using the view of matrix products in terms of linear combinations of rows, we have

$$A_1 = 1R_1 + 3R_2 + 2R_3 + 5R_4$$
$$A_3 = 2R_1 + 1R_2 + 1R_3 + 3R_4$$
$$A_4 = 5R_1 + 5R_2 + 4R_3 + 11R_4$$

We can then directly compute

$$A_1 + 2A_3 = (1R_1 + 3R_2 + 2R_3 + 5R_4) + 2(2R_1 + 1R_2 + 1R_3 + 3R_4)$$
$$= 5R_1 + 5R_2 + 4R_3 + 11R_4$$
$$= A_4$$

3. *(15 pts)* For the matrix M below, we would like to compute the inverse matrix and the determinant. Compute BOTH of these quantities specifically using a SINGLE row reduction in BOTH calculations.

$$M = \begin{pmatrix} 7 & 12 \\ 4 & 7 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 7 & 12 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 4 & 7 & 0 & 1 \\ 7 & 12 & 1 & 0 \end{array}\right) \begin{array}{l} ② \\ ① \end{array} \qquad \det = -\det M$$

$$\left(\begin{array}{cc|cc} 8 & 14 & 0 & 2 \\ 7 & 12 & 1 & 0 \end{array}\right) \begin{array}{l} 2① \\ ② \end{array} \qquad \det = -2\det M$$

$$\left(\begin{array}{cc|cc} 1 & 2 & -1 & 2 \\ 7 & 12 & 1 & 0 \end{array}\right) \begin{array}{l} ①-② \\ ② \end{array} \qquad \det = -2\det M$$

$$\left(\begin{array}{cc|cc} 1 & 2 & -1 & 2 \\ 0 & -2 & 8 & -14 \end{array}\right) \begin{array}{l} ① \\ ②-7① \end{array} \qquad \det = -2\det M$$

$$\left(\begin{array}{cc|cc} 1 & 2 & -1 & 2 \\ 0 & 1 & -4 & 7 \end{array}\right) \begin{array}{l} ① \\ ②/-2 \end{array} \qquad \det = \det M$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 7 & -12 \\ 0 & 1 & -4 & 7 \end{array}\right) \begin{array}{l} ①-2② \\ ② \end{array} \qquad \det = \det M$$

$$\underbrace{}_{R} \qquad \underbrace{}_{E}$$

We have $R = I$, so M is invertible and $\boxed{M^{-1} = E = \begin{pmatrix} 7 & -12 \\ -4 & 7 \end{pmatrix}}$

Tracking effects on det., we have $\det I = \det M$, so

$\boxed{\det M = 1}$.

4. *(15 pts)* The 4×4 matrix N has columns as indicated below, and has determinant equal to x.

$$N = \begin{pmatrix} | & | & | & | \\ \vec{a} & \vec{b} & \vec{c} & \vec{d} \\ | & | & | & | \end{pmatrix}$$

Compute the determinants of the matrices below, in terms of x.

(a) $P = \begin{pmatrix} | & | & | & | \\ \vec{c} & \vec{b} & \vec{a} & \vec{d} \\ | & | & | & | \end{pmatrix}$

This results from N by a row switch, so

$$\det P = -\det N = \boxed{-x}$$

two same cols

(b) $Q = \begin{pmatrix} | & | & | & | \\ (3\vec{a} + 2\vec{c}) & \vec{b} & \vec{c} & \vec{d} \\ | & | & | & | \end{pmatrix}$ $\det Q = 3 \det \begin{pmatrix} | & | & | & | \\ \vec{a} & \vec{b} & \vec{c} & \vec{d} \\ | & | & | & | \end{pmatrix} + 2 \det \begin{pmatrix} | & | & | & | \\ \vec{c} & \vec{b} & \vec{c} & \vec{d} \\ | & | & | & | \end{pmatrix}$

$= 3(x) + 2(0)$

$= \boxed{3x}$

(c) $R = \begin{pmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{c} & - \\ - & \vec{d} & - \end{pmatrix}$ $\det R = \det N^T = \det N = \boxed{x}$

5. *(20 pts)* Show that the pair of vectors $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}3\\5\\1\end{pmatrix}$ spans the plane with equation below.

$$3x - 2y + z = 0$$

Need to show we can always solve

$$c_1\begin{pmatrix}1\\2\\1\end{pmatrix} + c_2\begin{pmatrix}3\\5\\1\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\-3b_1+2b_2\end{pmatrix}$$

which is equivalent to the system reduced below.

$$\left(\begin{array}{cc|c}1 & 3 & b_1\\2 & 5 & b_2\\1 & 1 & -3b_1+2b_2\end{array}\right)$$

$$\left(\begin{array}{cc|c}1 & 3 & b_1\\0 & -1 & -2b_1+b_2\\0 & -2 & -4b_1+2b_2\end{array}\right)\begin{array}{l}1\\2-2(1)\\3-(1)\end{array}$$

$$\left(\begin{array}{cc|c}1 & 0 & -5b_1+3b_2\\0 & 1 & 2b_1-b_2\\0 & 0 & 0\end{array}\right)\begin{array}{l}1+3(2)\\-(2)\\3-2(2)\end{array}$$

There is no contradiction, so a solution always exists $(c_1 = -5b_1+3b_2,\ c_2 = 2b_1-b_2)$, so the vectors span the plane as desired.

6. *(15 pts)* Show that the set $V = \left\{ f \in C^0[0,1] \middle| \int_0^1 f(x)\,dx = 0 \right\}$ is a subspace of $C^0[0,1]$.

To confirm V is closed under addition, assume $f, g \in V$, so that $\int_0^1 f\,dx = 0$, $\int_0^1 g\,dx = 0$.

Then $\int_0^1 (f+g)\,dx = \int_0^1 f\,dx + \int_0^1 g\,dx = 0 + 0 = 0$

So $f+g \in V$ also, as required.

To confirm V is closed under scalar multiplication, assume $f \in V$, so that $\int_0^1 f\,dx = 0$

Then $\int_0^1 (cf)\,dx = c\int_0^1 f\,dx = c \cdot 0 = 0$

So $cf \in V$ also, as required.

The above two results show V is a subspace.