EXAM 3

Math 216, 2014-2015 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________ Day/Time________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ________

1. ________
2. ________
3. ________
4. ________
5. ________

Total Score ________ (/100 points)

1. *(20 pts)* Suppose that $\vec{v}$ is a complex eigenvector of the real matrix A, with complex eigenvalue $2 - 3i$. Find another eigenvector of A, and find its eigenvalue.

$\overline{\vec{v}}$ must also be an eigenvector of A.

$$A\overline{\vec{v}} = \overline{A}\,\overline{\vec{v}} = \overline{A\vec{v}} = \overline{(2-3i)\vec{v}} = (2+3i)\overline{\vec{v}}$$

b/c A is real

b/c $\vec{v}$ is an eigenvector

So the eigenvalue of $\overline{\vec{v}}$ is $2+3i$.

2. *(20 pts)* The matrix A is known to satisfy the following equation.

$$J \quad \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 7 & 1 & 7 & 2 & 1 & 4 \\ 2 & 6 & 3 & 2 & 1 & 1 & 2 \\ 3 & 5 & 2 & 3 & 3 & 2 & 5 \\ 4 & 4 & 6 & 1 & 3 & 6 & 8 \\ 5 & 3 & 7 & 9 & 3 & 5 & 2 \\ 6 & 2 & 4 & 4 & 6 & 4 & 1 \\ 7 & 1 & 1 & 3 & 7 & 2 & 0 \end{pmatrix}^{-1}}^{P^{-1}} \begin{pmatrix} \\ \\ \\ & A & \\ \\ \\ \\ \end{pmatrix} \overbrace{\begin{pmatrix} 1 & 7 & 1 & 7 & 2 & 1 & 4 \\ 2 & 6 & 3 & 2 & 1 & 1 & 2 \\ 3 & 5 & 2 & 3 & 3 & 2 & 5 \\ 4 & 4 & 6 & 1 & 3 & 6 & 8 \\ 5 & 3 & 7 & 9 & 3 & 5 & 2 \\ 6 & 2 & 4 & 4 & 6 & 4 & 1 \\ 7 & 1 & 1 & 3 & 7 & 2 & 0 \end{pmatrix}}^{P}$$

$\vec{v}_1$ $\vec{v}_2$ $\vec{v}_2$

(a) Compute the product of the matrix A and the vector $\vec{v}_2 = (7, 6, 5, 4, 3, 2, 1)$.

$J = P^{-1}AP \Rightarrow PJ = AP$

$A\vec{v}_2$ is 2nd column of AP, so also of PJ, which is

$1\vec{v}_1 + 3\vec{v}_2 = (22 \quad 20 \quad 18 \quad 16 \quad 14 \quad 12 \quad 10)$

(b) Find all of the eigenvectors of A.

These are the column vectors of P corresponding to the locations of the first columns of each basic Jordan block.

$\vec{v}_1 = (1 \; 2 \; 3 \; 4 \; 5 \; 6 \; 7) \leftarrow$ eval $= 3$

$\vec{v}_3 = (1 \; 3 \; 2 \; 6 \; 7 \; 4 \; 1) \leftarrow$ eval $= 2$

$\vec{v}_5 = (2 \; 1 \; 3 \; 3 \; 3 \; 6 \; 7) \leftarrow$ eval $= 6$

(c) Your friend Bob says he thinks $(12, 1, 1, 1, 4, 5, 6)$ is an eigenvector of A. Given only the information above, can it be determined if Bob is right or wrong, or is there not enough information given? Explain.

Bob is definitely wrong. Each eigenspace is one dimensional, so any eigenvector would have to be a scalar multiple of one of the three noted eigenvectors. Bob's vector isn't.

3. *(20 pts)* Recall that the angle addition formulas

$$\begin{aligned}\cos(a+b) &= \cos a\cos b - \sin a \sin b\\ \sin(a+b) &= \sin a \cos b + \cos a \sin b\end{aligned}$$

can be used to show $\cos a \cos b = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$, and similar results.

(a) Show that, when $n \neq m$ are positive integers, the functions $\cos nx$ and $\cos mx$ are orthogonal by the L^2 inner product on the interval $[0, 2\pi]$.

$$\langle \cos nx, \cos mx\rangle = \int_0^{2\pi} \cos nx \cos mx \, dx$$

$$= \int_0^{2\pi} \frac{1}{2}\cos((n+m)x)\,dx + \int_0^{2\pi}\frac{1}{2}\cos((n-m)x)\,dx$$

$$= \frac{1}{2(n+m)}\sin((n+m)x)\Big]_0^{2\pi} + \frac{1}{2(n-m)}\sin((n-m)x)\Big]_0^{2\pi}$$

$$= (0-0) + (0-0) = 0$$

So these functions are orthogonal.

(b) Show that $\sin nx$ and $\cos mx$ are orthogonal using the same inner product.

$$\sin((n+m)x) = \sin nx \cos mx + \cos nx \sin mx$$

$$\sin((n-m)x) = \sin nx \cos mx - \cos nx \sin mx$$

$$\frac{1}{2}\left(\sin((n+m)x) + \sin((n-m)x)\right) = \sin nx \cos mx$$

So $\langle \sin nx, \cos mx\rangle = \int_0^{2\pi} \sin nx \cos mx \, dx$

$$= \int_0^{2\pi}\frac{1}{2}\sin((n+m)x)\,dx + \int_0^{2\pi}\frac{1}{2}\sin((n-m)x)\,dx$$

$$= \frac{-1}{2(n+m)}\cos((n+m)x)\Big]_0^{2\pi} + \frac{-1}{2(n-m)}\cos((n-m)x)\Big]_0^{2\pi}$$

$$= 0 + 0 = 0$$

So these functions are orthogonal.

4. *(20 pts)* The matrix A is known to satisfy the following equation.

$$\left(\quad A \quad \right) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} \boxed{4} & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

Use the given information to find a fundamental set of solutions for the system $\vec{y}' = A\vec{y}$.

$A = Q^{-1} J Q$, but $A = PJP^{-1}$ where P is the matrix whose columns are the Jordan basis. So $P = Q^{-1}$, which we compute by

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right) \begin{array}{l} ① - ③ \\ ② - 3③ \\ ③/2 \end{array} \Rightarrow P = \left(\underset{\vec{v}_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} \underset{\vec{v}_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \underset{\vec{v}_3}{\begin{pmatrix} -1 \\ -3 \\ 1/2 \end{pmatrix}} \right)$$

A fundamental set of solutions is formed by:

$$e^{xA}\vec{v}_1 = e^{4x}\vec{v}_1$$

$$e^{xA}\vec{v}_2 = e^{2x}\vec{v}_2$$

$$e^{xA}\vec{v}_3 = e^{2x}(\vec{v}_3 + x\vec{v}_2)$$

5. *(20 pts)* Find a particular solution to the system of equations below.

$$y_1' = 2y_1 + 3y_2 + 2x^2$$
$$y_2' = 3y_1 + 5y_2 + 3x - 1$$

$A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$, $\vec{y}' = A\vec{y} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} x^2 + \begin{pmatrix} 0 \\ 3 \end{pmatrix} x + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Guess $\vec{y}_p = \vec{a}x^2 + \vec{b}x + \vec{c}$; then equation becomes

$$2\vec{a}x + \vec{b} = A(\vec{a}x^2 + \vec{b}x + \vec{c}) + \begin{pmatrix} 2 \\ 0 \end{pmatrix} x^2 + \begin{pmatrix} 0 \\ 3 \end{pmatrix} x + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Grouping like powers of x and using $A^{-1} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$,

<u>x^2 terms</u>: $A\vec{a} = -\begin{pmatrix} 2 \\ 0 \end{pmatrix} \Rightarrow \vec{a} = A^{-1}\begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 \\ 6 \end{pmatrix}$

<u>x terms</u>: $A\vec{b} = 2\vec{a} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -20 \\ 9 \end{pmatrix}$

$$\Rightarrow \vec{b} = A^{-1}\begin{pmatrix} -20 \\ 9 \end{pmatrix} = \begin{pmatrix} -127 \\ 78 \end{pmatrix}$$

<u>const terms</u>: $A\vec{c} = \vec{b} - \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -127 \\ 79 \end{pmatrix}$

$$\Rightarrow \vec{c} = A^{-1}\begin{pmatrix} -127 \\ 79 \end{pmatrix} = \begin{pmatrix} -635 - 237 \\ 378 + 158 \end{pmatrix}$$

$$= \begin{pmatrix} -872 \\ 536 \end{pmatrix}$$

So $\vec{y}_p = \begin{pmatrix} -10 \\ 6 \end{pmatrix} x^2 + \begin{pmatrix} -127 \\ 78 \end{pmatrix} x + \begin{pmatrix} -872 \\ 536 \end{pmatrix}$