EXAM 2

Math 216, 2014-2015 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number__________ TA____________________ Day/Time____________________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ____________________

1. __________

2. __________

3. __________

4. __________

5. __________

Total Score __________ (/100 points)

1. (20pts) The differential equation $L(y) = 0$ has characteristic polynomial

$$p(\lambda) = (\lambda^2 + 3\lambda + 2)(\lambda^2 - 2\lambda + 2)^2$$

Find a fundamental set of real solutions to this differential equation.

$\lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$

$\lambda^2 - 2\lambda + 2$ has roots $\frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

So $p(\lambda)$ has roots $-1, -2$ of multiplicity 1
and $1 \pm i$ of multiplicity 2

By theorems from class, a fundamental set of real solutions is

$$\left\{ e^{-x}, e^{-2x}, e^{x}\cos x, e^{x}\sin x, xe^{x}\cos x, xe^{x}\sin x \right\}$$

2. (20pts) The function $T : C^0 \to \mathbb{R}$ is computed by

$$T(f) = \int_0^1 f(x)e^x \, dx$$

Prove or disprove: T is a linear transformation.

$$T(c_1f_1 + c_2f_2) = \int_0^1 (c_1f_1 + c_2f_2)e^x \, dx$$

$$= \int_0^1 c_1f_1e^x + c_2f_2e^x \, dx$$

$$= \int_0^1 c_1f_1e^x \, dx + \int_0^1 c_2f_2e^x \, dx$$

$$= c_1\int_0^1 f_1e^x \, dx + c_2\int_0^1 f_2e^x \, dx$$

$$= c_1T(f_1) + c_2T(f_2)$$

So T is a linear transformation.

3. (20pts) Recall that P_2 is the vector space of polynomials of degree at most 2. Use the Wronskian as part of your direct demonstration that the dimension of P_2 is 3.

We will show that $\{x^2, x, 1\}$ is a basis for P_2.

Span: $ax^2+bx+c \in P_2$ is a l.c. of $x^2, x, 1$ with coefficients a, b, c:

$$(ax^2+bx+c) = a(x^2) + b(x) + c(1)$$

Independence: The Wronskian is

$$W(x) = \det\begin{pmatrix} x^2 & x & 1 \\ 2x & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$W(0) = \det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} = -2 \neq 0$$

This is a nonzero value of $W(x)$, so the list is independent.

These two results show $\{x^2, x, 1\}$ is a basis, and thus $\dim(P_2) = 3$.

4. (20pts)

(a) Apply the linear transformation $(D-3)$ directly to find a general formula for $(D-3)(x^k e^{3x})$, where k is a positive integer.

$$(D-3)(x^k e^{3x}) = (x^k e^{3x})' - 3(x^k e^{3x})$$
$$= k x^{k-1} e^{3x} + x^k (3e^{3x}) - 3x^k e^{3x}$$
$$= k x^{k-1} e^{3x}$$

(b) Use the above result to show directly that $y = x^4 e^{3x}$ is a solution to $L(y) = 0$, whose characteristic polynomial is

$$p(\lambda) = (\lambda - 4)^3 (\lambda - 3)^5 (\lambda^2 + 2\lambda - 1)^3$$

$$L(y) = p(D)(y)$$
$$= (D-4)^3 (D-3)^5 (D^2+2D-1)^3 (x^4 e^{3x})$$
$$= \underbrace{(D-4)^3 (D^2+2D-1)^3}_{Q(D)} (D-3)(D-3)^4 (x^4 e^{3x})$$
$$= Q(D) \quad (D-3)(4 \cdot 3 \cdot 2 \cdot 1\, e^{3x})$$
$$= Q(D) \left((24 e^{3x})' - 3(24 e^{3x}) \right)$$
$$= Q(D)(0)$$
$$= 0$$

5. (20pts)

(a) Find a particular solution to the differential equation

$$z''(t) + z'(t) - z(t) = 7e^{2it}$$

Guess: $z = T \cdot 7e^{2it}$. Then the DE becomes

$$\left(-4 \cdot T \cdot 7e^{2it}\right) + \left(2i \cdot T \cdot 7e^{2it}\right) - \left(T \cdot 7e^{2it}\right) = 7e^{2it}$$

$$-4T + 2iT - T = 1$$

$$T = \frac{1}{-5+2i} = \frac{-5-2i}{29}$$

So $z = \frac{-35-14i}{29} e^{2it} = \left(\frac{-35-14i}{29}\right)\left(\cos 2t + i \sin 2t\right)$

$$= \left(\frac{-35}{29}\cos 2t + \frac{14}{29}\sin 2t\right) + i\left(\frac{-14}{29}\cos 2t - \frac{35}{29}\sin 2t\right)$$

(b) Find solutions to each of the differential equations below.

$$\begin{aligned} u''(t) + u'(t) - u(t) &= 7\cos(2t) \\ v''(t) + v'(t) - v(t) &= 7\sin(2t) \end{aligned}$$

$7\cos 2t = \mathrm{Re}\left(7e^{2it}\right)$

So $u = \mathrm{Re}(z) = \frac{-35}{29}\cos 2t + \frac{14}{29}\sin 2t$

$7\sin 2t = \mathrm{Im}\left(7e^{2it}\right)$

So $v = \mathrm{Im}(z) = \frac{-14}{29}\cos 2t - \frac{35}{29}\sin 2t$