EXAM 1

Math 216, 2014-2015 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________ Day/Time________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________

1. ________

2. ________

3. ________

4. ________

5. ________

6. ________

Total Score ________ (/100 points)

1. *(24 pts)* The matrix $[A|I]$ is row equivalent to

$$\left(\begin{array}{cccc|ccc} 1 & 0 & 2 & 1 & 1 & 2 & 4 \\ 2 & 1 & 3 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 3 & 1 \end{array}\right)$$

(a) Find the reduced row echelon form (R) of A, and the row reduction matrix (E) that reduces A to R.

$$\left(\begin{array}{cccc|ccc} 1 & 0 & 2 & 1 & 1 & 2 & 4 \\ 0 & 1 & -1 & -2 & -1 & -3 & -6 \\ 0 & 0 & 0 & 0 & 2 & 3 & 1 \end{array}\right) \begin{array}{l} ① \\ ② - 2① \\ ③ \end{array}$$

$$\underbrace{}_{R} \quad \underbrace{}_{E}$$

(b) Decide if $A\vec{x} = \vec{b}$ has a solution and how many solutions there are, when $\vec{b} = (2, -1, -1)$; and likewise when $\vec{b} = (1, 0, 0)$.

$$A\vec{x} = \vec{b} \iff R\vec{x} = E\vec{b}$$

$\vec{b} = (2,-1,-1)$:

$$R\vec{x} = \begin{pmatrix} -4 \\ 7 \\ 0 \end{pmatrix}$$

no contr., free vars $\Rightarrow \infty$ sols

$\vec{b} = (1,0,0)$:

$$R\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

contradiction, so no sols.

(c) Your friend Bob says that the determinant of E is 5, and thus that the determinant of A must be 1/5. How would you respond to Bob most helpfully?

A is not square, so there is no determinant.

2. *(16 pts)* Find the matrix B such that the rows of BA are related to the rows of A as indicated below.

$$\begin{pmatrix} & & \\ & B & \\ & & \end{pmatrix}\begin{pmatrix} — & A_1 & — \\ — & A_2 & — \\ — & A_3 & — \end{pmatrix} = \begin{pmatrix} — & 3A_3 - A_2 & — \\ — & 2A_1 + A_3 & — \\ — & A_1 - A_2 & — \end{pmatrix}$$

$(BA)_1 = (0)A_1 + (-1)A_2 + (3)A_3$

So $B_1 = (0 \ \ -1 \ \ 3)$

$(BA)_2 = (2)A_1 + (0)A_2 + (1)A_3$

So $B_2 = (2 \ \ 0 \ \ 1)$

$(BA)_3 = (1)A_1 + (-1)A_2 + (0)A_3$

So $B_3 = (1 \ \ -1 \ \ 0)$

Then $B = \begin{pmatrix} 0 & -1 & 3 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

3. *(20 pts)* The square matrix A has the feature that its three first rows A_1, A_2, A_3 are related by

$$2A_1 - 7A_2 + 5A_3 = \vec{0}$$

Show that the third column of A, and the third column in the cofactor matrix of A, are perpendicular (both being viewed as vectors in $\mathbb{R}^n$).

The given relation shows $A_1 = \left(\frac{7}{2}\right)A_2 - \left(\frac{5}{2}\right)A_3$, so the row operations of adding opposite multiples of A_2, A_3 to the first row will result in a matrix with determinant zero; and thus $\det A = 0$.

Alternatively $\det A$ can be computed as a third column cofactor expansion $\sum_{i=1}^{n} a_{i3} C_{i3}$, which is the dot product of the third column of A and the third column of C.

So this dot product is zero, and these columns are perpendicular viewed as vectors.

4. *(20 pts)* Decide if the columns of the product matrix B below are linearly independent or linearly dependent.

$$B = \begin{pmatrix} | & | & | \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ | & | & | \end{pmatrix} = \begin{array}{c} \begin{pmatrix} e & 12 & 2 \\ \pi & 234 & 3\pi \\ \sqrt{134} & \sqrt[3]{21} & e^2 \end{pmatrix} \\ \begin{array}{ccc} \vec{v}_1 & \quad V_2 & \quad V_3 \end{array} \end{array} \begin{pmatrix} 2 & 10 & 12 \\ 1 & 12 & 13 \\ 3 & 14 & 17 \end{pmatrix}$$

$$\vec{b}_1 = 2\vec{v}_1 + 1\vec{v}_2 + 3\vec{v}_3$$

$$\vec{b}_2 = 10\vec{v}_1 + 12\vec{v}_2 + 14\vec{v}_3$$

$$\vec{b}_3 = 12\vec{v}_1 + 13\vec{v}_2 + 17\vec{v}_3$$

So $\vec{b}_3 = \vec{b}_1 + \vec{b}_2$, and thus the columns of B are linearly dependent.

5. *(20 pts)* In this question we consider the set W of all real-valued functions on $\mathbb{R}$ for which $|f(0)| \leq |f(1)|$.

(a) Is W closed under addition? Prove or find a counterexample.

W is <u>not</u> closed under addition. Counterexample:

$f(x) = 1+x$: $f(0)=1$, $f(1)=2$, and $|1| \leq |2|$ ✓

$g(x) = 1-3x$: $g(0)=1$, $g(1)=-2$, and $|1| \leq |-2|$ ✓

But $(f+g)(x) = 2-2x$: $(f+g)(0)=2$, $(f+g)(1)=0$, and $|2| \not\leq |0|$

So $f, g \in W$, but $f+g \notin W$.

(b) Is W closed under scalar multiplication? Prove or find a counterexample.

Yes. Suppose $f \in W$, and thus $|f(0)| \leq |f(1)|$

Then
$$\begin{aligned} |(cf)(0)| &= |c\, f(0)| = |c|\,|f(0)| \\ &\leq |c|\,|f(1)| \quad (\text{because } |c| \geq 0) \\ &\leq |c\, f(1)| \\ &\leq |(cf)(1)| \end{aligned}$$

So $(cf) \in W$. Thus W is closed under scalar mult.