EXAM 3

Math 216, 2013-2014 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions ________________

Disc.: Number__________ TA____________________ Day/Time____________________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ____________________

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

Total Score __________ (/100 points)

1. *(20 pts)* The Jordan form for the matrix $A = [T]_{\mathcal{S}}^{\mathcal{S}}$ is $J = [T]_{\mathcal{V}}^{\mathcal{V}} = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}$, where $\mathcal{S}$ is the standard basis and $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, ordered as listed.

(a) What can we conclude about $T(\vec{v}_2)$?

$$\boxed{T(\vec{v}_2) = 1\vec{v}_1 + 5\vec{v}_2 + 0\vec{v}_3}$$

(b) What can you conclude about the eigenvectors of A?

$T(\vec{v}_1) = 5v_1$ $\qquad$ $T(\vec{v}_3) = 8\vec{v}_3$

So $\quad A\underbrace{[\vec{v}_1]_{\mathcal{S}}} = 5[\vec{v}_1]_{\mathcal{S}}$ $\qquad$ $A\underbrace{[\vec{v}_3]_{\mathcal{S}}} = 8[\vec{v}_3]_{\mathcal{S}}$

These are the eigenvectors of A.

(c) Find another Jordan form matrix J_2 that is similar to the above matrix J, and a basis $\mathcal{W}$ with $J_2 = [T]_{\mathcal{W}}^{\mathcal{W}}$.

Choosing $\boxed{\mathcal{W} = \{\vec{v}_3, \vec{v}_1, \vec{v}_2\}}$, the same statements about $T(\vec{v}_i)$ are expressed by

$$[T]_{\mathcal{W}}^{\mathcal{W}} = \boxed{\begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix} = J_2}$$

(d) Suppose the standard basis representations of the vectors in $\mathcal{V}$ are $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$, respectively. Compute $e^{xA}\vec{v}_2$.

$$e^{xA}\vec{v}_2 = e^{5x}(\vec{v}_2 + x\vec{v}_1) = \boxed{e^{5x}\begin{pmatrix} 1+x \\ 0+3x \\ 0+2x \end{pmatrix}}$$

2. *(20 pts)* In this problem we will use the "Hermitian norm", defined by $\|\vec{w}\|_H = \sqrt{\langle \vec{w}, \vec{w} \rangle_H}$. We consider the vectors

$$\vec{w}_1 = \begin{pmatrix} 1+i \\ 3+3i \\ 4i \end{pmatrix} \quad \text{and} \quad \vec{w}_2 = \begin{pmatrix} 1-i \\ 1+i \\ 2i \end{pmatrix}$$

(a) Compute $\|\vec{w}_1\|_H$, and $\vec{v}_1 = \vec{w}_1 / \|\vec{w}_1\|_H$.

$$\|\vec{w}_1\|_H = \sqrt{(1+i)\overline{(1+i)} + (3+3i)\overline{(3+3i)} + (4i)\overline{(4i)}} = \sqrt{2+18+16} = \boxed{6}$$

$$\vec{v}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|_H} = \begin{pmatrix} 1+i \\ 3+3i \\ 4i \end{pmatrix} \Big/ 6 = \boxed{\begin{pmatrix} (1+i)/6 \\ (1+i)/2 \\ 2i/3 \end{pmatrix}}$$

(b) Compute $\langle \vec{v}_1, \vec{w}_2 \rangle_H$.

$$= \frac{1}{6}\left((1+i)\overline{(1-i)} + (3+3i)\overline{(1+i)} + (4i)\overline{(2i)} \right)$$

$$= \frac{1}{6}\left(2i + 6 + 8\right) = \boxed{\frac{7+i}{3}}$$

(c) Use properties of the Hermitian dot product (be careful with conjugates!) to show that $\left\langle \vec{v}_1, \vec{w}_2 - \left(\langle \vec{v}_1, \vec{w}_2 \rangle_H\right) \vec{v}_1 \right\rangle_H = 0.$

$$\parallel$$

$$\langle \vec{v}_1, \vec{w}_2 \rangle_H - \left\langle \vec{v}_1, \left(\langle \vec{v}_1, \vec{w}_2 \rangle_H\right) \vec{v}_1 \right\rangle_H$$

these cancel!

$$= \langle \vec{v}_1, \vec{w}_2 \rangle_H - \left(\overline{\overline{\langle \vec{v}_1, \vec{w}_2 \rangle_H}}\right) \underbrace{\langle \vec{v}_1, \vec{v}_1 \rangle_H}_{=1}$$

$$= \langle \vec{v}_1, \vec{w}_2 \rangle_H - \langle \vec{v}_1, \vec{w}_2 \rangle_H = 0$$

(d) Use the above ideas to find a vector $\vec{v}_2$ so that $\|\vec{v}_2\|_H = 1$ and $\langle \vec{v}_1, \vec{v}_2 \rangle_H = 0$.

Calling this, $\vec{X}_2$, we choose $\boxed{\vec{v}_2 = \frac{\vec{X}_2}{\|\vec{X}_2\|}}$. Then

$$\|\vec{v}_2\|_H = \left\| \frac{\vec{X}_2}{\|\vec{X}_2\|_H} \right\|_H = \frac{1}{\|\vec{X}_2\|_H} \|\vec{X}_2\|_H = 1 \quad \text{and} \quad \langle \vec{v}_1, \vec{v}_2 \rangle_H = \left\langle \vec{v}_1, \frac{\vec{X}_2}{\|\vec{X}_2\|_H} \right\rangle_H$$

$$= \frac{1}{\|\vec{X}_2\|_H} \langle \vec{v}_1, \vec{X}_2 \rangle = 0$$

3. *(20 pts)* Find a fundamental set of solutions to the system below.

$$\begin{aligned} y_1' &= 7y_1 - 6y_2 \\ y_2' &= 2y_1 \end{aligned}$$

$\vec{y}' = A\vec{y}, \; A = \begin{pmatrix} 7 & -6 \\ 2 & 0 \end{pmatrix}$

$$p(\lambda) = \det(A - \lambda I) = (7-\lambda)(0-\lambda) - (-6)(2) = \lambda^2 - 7\lambda + 12$$
$$= (\lambda - 3)(\lambda - 4)$$

Eigenvalues are 3 and 4.

For $\lambda = 3$: $A - \lambda I = \begin{pmatrix} 4 & -6 \\ 2 & -3 \end{pmatrix}$ row reduces to $\begin{pmatrix} 1 & -3/2 \\ 0 & 0 \end{pmatrix}$

giving eigenvector $\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

For $\lambda = 4$: $A - \lambda I = \begin{pmatrix} 3 & -6 \\ 2 & -4 \end{pmatrix}$ row reduces to $\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$

giving eigenvector $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

F.S.S. is given by $\left\{ e^{x\lambda_1} \vec{v}_1, e^{x\lambda_2} \vec{v}_2 \right\}$

$$= \boxed{\left\{ e^{3x} \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \; e^{4x} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}}$$

4. *(20 pts)* The matrix A below has characteristic polynomial $p(\lambda) = (\lambda - 3)^3$.

$$A = \begin{pmatrix} 7 & -3 & -1 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{pmatrix}$$

Find a Jordan basis for A. You might find it useful to know that the matrix E below represents a row reduction for the matrix $A - 3I$.

$$E = \begin{pmatrix} -2 & 0 & 3 \\ -3 & 0 & 4 \\ 1 & -1 & 0 \end{pmatrix}$$

$M = A - 3I = \begin{pmatrix} 4 & -3 & -1 \\ 4 & -3 & -1 \\ 3 & -2 & -1 \end{pmatrix}$. We reduce $(M|I)$ by

$$E(M|I) = \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 0 & 3 \\ 0 & 1 & -1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right) = (R|E)$$

Eigenvector(s): Solve $M\vec{v} = \vec{0}$ by $R\vec{v} = E\vec{0} = \vec{0}$.

$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix}$; choosing $z=1$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Next Jordan vector: Solve $M\vec{v}_2 = \vec{v}_1$ by $R\vec{v} = E\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+z \\ 1+z \\ z \end{pmatrix}$; choosing $z=0$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Next Jordan vector: Solve $M\vec{v}_3 = \vec{v}_2$ by $R\vec{v} = E\vec{v}_2 = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}$

$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z-2 \\ z-3 \\ z \end{pmatrix}$; choosing $z=0$, $\vec{v}_3 = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}$

So $\boxed{\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} \right\}}$ is a Jordan basis

5. For the following question you will make use of this theorem:

Theorem: If T_1, T_2, T_3 are linear transformations from V to $\mathbb{R}$, then the function $T : V \to \mathbb{R}^3$ defined by $T(v) = \begin{pmatrix} T_1(v) \\ T_2(v) \\ T_3(v) \end{pmatrix}$ is also a linear transformation.

Your friend Bob is considering three functions $f_1, f_2, f_3 \in C^1[0,1]$. He knows only the following information about these three functions:

1. Their values at $x = 0$ are, respectively, 1, 2, 3.
2. The values of their derivatives at $x = 0.5$ are, respectively, 5, 7, 2.
3. The integrals of these functions from 0.3 to 0.9 are, respectively, 0, 2, 5.

Bob needs to know if this trio of functions is linearly independent or linearly dependent, but he is frustrated by the fact that he does not have enough information to use the Wronskian.

In using the above theorem to help Bob come to a conclusion,

(a) *(6 pts)* identify your choices of linear transformations T_1, T_2, T_3

(b) *(6 pts)* find the values of $T(f_1), T(f_2), T(f_3)$

(c) *(8 pts)* explain what you can conclude about the three vectors in (b), and how this allows you to draw a conclusion about the original three functions.

a) We choose $\boxed{\begin{array}{l} T_1(f) = f(0) \\ T_2(f) = f'(0.5) \\ T_3(f) = \int_{0.3}^{0.9} f\,dx \end{array}}$ So $T(f) = \begin{pmatrix} f(0) \\ f'(0.5) \\ \int_{0.3}^{0.9} f\,dx \end{pmatrix}$

b) The given info then means

$$\boxed{T(f_1) = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}, \quad T(f_2) = \begin{pmatrix} 2 \\ 7 \\ 2 \end{pmatrix}, \quad T(f_3) = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}}$$

c) $\det\begin{pmatrix} 1 & 2 & 3 \\ 5 & 7 & 2 \\ 0 & 2 & 5 \end{pmatrix} = 1(31) - 5(4) + 0(-17) = 11 \neq 0$,

so $\{T(f_1), T(f_2), T(f_3)\}$ is linearly independent.

Because these images are independent and T is a linear transformation, we know $\{f_1, f_2, f_3\}$ is also l. i.