EXAM 2

Math 216, 2013-2014 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________________ Day/Time________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. ________

2. ________

3. ________

4. ________

5. ________

6. ________

Total Score ________ (/100 points)

1. *(16 pts)*

(a) The Wronskian of the functions $f_1(x), f_2(x), f_3(x)$ is $w(x) = \sin x$. What if anything can you conclude from this about the independence or dependence of this list of functions?

At at least one point the Wronskian is non-zero, so the list must be independent.

(b) The Wronskian of the functions $g_1(x), g_2(x), g_3(x)$ is $w(x) = 0$. What if anything can you conclude from this about the independence or dependence of this list of functions?

Nothing.

(c) For which of the above two lists would it be possible to draw a stronger conclusion if you knew that the functions in that list were analytic?

If the list in part (b) were all analytic functions, then we could conclude the list was dependent.

(d) The Wronskian of the functions $y_1(x), y_2(x), y_3(x)$ is $w(x) = 0$, and these functions are also known to be solutions to the differential equation $(x^2+1)y'''' - x^2y'' + y = 0$. What if anything can you conclude from this about the independence or dependence of this list of functions?

Nothing. The theorem from section 4.1 does not apply because the number of functions (3) does not equal the order of the DE (4).

2. *(18 pts)* Find a fundamental set of solutions for the differential equation below.

$$y''' - y'' - 5y' - 3y = 0$$

$p(\lambda) = \lambda^3 - \lambda^2 - 5\lambda - 3$, poss. rat'l roots are $\pm 1, \pm 3$.

$p(1) = -8 \neq 0$

$p(-1) = 0 \Rightarrow (+1)$ is a factor

$$\begin{array}{r|l} & \lambda^2 - 2\lambda - 3 \\ \lambda+1 & \lambda^3 - \lambda^2 - 5\lambda - 3 \\ & \underline{\lambda^3 + \lambda^2} \\ & -2\lambda^2 - 5\lambda - 3 \\ & \underline{-2\lambda^2 - 2\lambda} \\ & -3\lambda - 3 \\ & \underline{-3\lambda - 3} \\ & 0 \end{array}$$

$\Rightarrow p(\lambda) = (\lambda+1)(\lambda^2 - 2\lambda - 3)$

$= (\lambda+1)(\lambda+1)(\lambda-3)$

Roots are $r_1 = -1 \ (m_1 = 2)$

$r_2 = 3 \ (m_2 = 1)$

By theorems from class then, a f.s.s. is

$$\{ e^{-x}, xe^{-x}, e^{3x} \}$$

3. *(16 pts)* Find a fundamental set of solutions for the differential equation below.

$$y'''' - 6y''' + 18y'' - 30y' + 25y = 0$$

(Hint: The complex number $z = 1 - 2i$ is a root of the characteristic polynomial.)

$1+2i$ must also be a root, and thus a quadratic factor is

$$(\lambda - (1-2i))(\lambda - (1+2i)) = \lambda^2 - 2\lambda + 5$$

$$\begin{array}{r|l}
 & \lambda^2 - 4\lambda + 5 \\
\lambda^2 - 2\lambda + 5 & \lambda^4 - 6\lambda^3 + 18\lambda^2 - 30\lambda + 25 \\
 & \underline{\lambda^4 - 2\lambda^3 + 5\lambda^2} \\
 & -4\lambda^3 + 13\lambda^2 - 30\lambda + 25 \\
 & \underline{-4\lambda^3 + 8\lambda^2 - 20\lambda} \\
 & 5\lambda^2 - 10\lambda + 25 \\
 & \underline{5\lambda^2 - 10\lambda + 25} \\
 & 0
\end{array}$$

Then $p(\lambda) = (\lambda - (1-2i))(\lambda - (1+2i))(\lambda^2 - 4\lambda + 5)$

from quadratic equation

$= (\lambda - (1-2i))(\lambda - (1+2i))(\lambda - (2+i))(\lambda - (2-i))$

By theorems from class then, a f.s.s. is

$$\{e^x \cos 2x,\ e^x \sin 2x,\ e^{2x} \cos x,\ e^{2x} \sin x\}$$

4. *(16 pts)* Write the form (but do not compute the undetermined coefficients) of a particular solution to the differential equation below.

$$p(\lambda) = \lambda^2 - 4\lambda + 5 \qquad y'' - 4y' + 5y = \underbrace{x^2 e^{2x}\cos x}_{g_1} - \underbrace{x e^{3x}\sin 4x}_{g_2}$$

$$= (\lambda-(2-i))(\lambda-(2+i))$$

For g_1: $r_1 = 2+i$ is a root of $p(\lambda)$, with $m=1$

Guess is $$Y_1 = x^1(c_2x^2 + c_1x + c_0)e^{2x}\cos x + x^1(d_2x^2 + d_1x + d_0)e^{2x}\sin x$$

For g_2: $r_2 = 3+4i$ is not a root of $p(\lambda)$

Guess is $$Y_2 = (e_1x + e_0)e^{3x}\cos 4x + (f_1x + f_0)e^{3x}\sin 4x$$

Total guess is:

$Y_p = Y_1 + Y_2$

5. Bob is attempting to prove a theorem discussed in class in Math 216, that the vector space of solutions to an nth order linear homogeneous differential equation $L(y) = 0$ satisfying the conditions of the existence/uniqueness theorem (we will call this space V) has dimension equal to n. To this end he considers the linear transformation $S : V \to \mathbb{R}^n$ defined by

$$S(y) = \begin{pmatrix} y(0) \\ y'(0) \\ \vdots \\ y^{[n-1]}(0) \end{pmatrix}$$

(For a solution y, the value of this linear transformation is the vector of initial values for that solution at $x_0 = 0$.)

(a) *(12 pts)* Prove that this is a linear transformation.

$$S(f+g) = \begin{pmatrix} (f+g)(0) \\ \vdots \\ (f+g)^{[n-1]}(0) \end{pmatrix} = \begin{pmatrix} f(0) + g(0) \\ \vdots \\ f^{[n-1]}(0) + g^{[n-1]}(0) \end{pmatrix}$$

$$= \begin{pmatrix} f(0) \\ \vdots \\ f^{[n-1]}(0) \end{pmatrix} + \begin{pmatrix} g(0) \\ \vdots \\ g^{[n-1]}(0) \end{pmatrix} = S(f) + S(g)$$

$$S(cf) = \begin{pmatrix} (cf)(0) \\ \vdots \\ (cf)^{[n-1]}(0) \end{pmatrix} = \begin{pmatrix} c\,f(0) \\ \vdots \\ c\,f^{[n-1]}(0) \end{pmatrix} = c \begin{pmatrix} f(0) \\ \vdots \\ f^{[n-1]}(0) \end{pmatrix} = c\,S(f)$$

So S is a l.t.

(b) *(3 pts)* Bob needs to know that this linear transformation is one-to-one – that is, that if $S(f) = S(g)$, then it must be that $f = g$. Specifically which of the assumptions in the statement allow(s) Bob to correctly draw this conclusion?

Here, one-to-one means that for an initial condition there can be at most one solution to the LDE. This comes from the assumption that we can invoke the existence/uniqueness thm.

(c) *(3 pts)* Bob needs to know that this linear transformation is onto – that is, that for every vector $\vec{k} \in \mathbb{R}^n$, there is a vector $f \in V$ such that $S(f) = \vec{k}$. Specifically which of the assumptions in the statement allow(s) Bob to correctly draw this conclusion?

Here, onto means that for every initial condition there must be at least one solution to the LDE. This too comes from the assumption that we can invoke the existence/uniqueness thm.

6. *(16 pts)* S is the vector space with basis $\alpha = \{v_1, v_2\}$, where $v_1 = \sin(\omega t)$ and $v_2 = \cos(\omega t)$. Consider the linear transformation $h : S \to S$ defined by

$$h(c_1 \sin(\omega t) + c_2 \cos(\omega t)) = 4c_1 \sin(\omega t - \pi/2) + 4c_2 \cos(\omega t - \pi/2)$$

(This linear transformation models the behavior of a damped harmonic oscillator with sinusoidal external force for which the gain equals 4 and phase shift equals to $\pi/2$.)

We will consider also the basis $\beta = \{w_1, w_2\}$, where $w_1 = \sin(\omega t - \pi/6)$ and $w_2 = \cos(\omega t - \pi/6)$, and the vector $x = \sin(\omega t) + 2\cos(\omega t)$.

(a) Compute $[h]_\alpha^\alpha$.

$$h(v_1) = h(\sin \omega t) = 4\sin(\omega t - \pi/2) = 0 \sin \omega t - 4 \cos \omega t = {}_\alpha\begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

$$h(v_2) = h(\cos \omega t) = 4\cos(\omega t - \pi/2) = 4 \sin \omega t + 0 \cos \omega t = {}_\alpha\begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$[h]_\alpha^\alpha = \Big[\, [h(v_1)]_\alpha \;\; [h(v_2)]_\alpha \,\Big] = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

(b) Confirm by direct calculation that $[h]_\alpha^\alpha [x]_\alpha = [h(x)]_\alpha$.

$$h(x) = 4\sin(\omega t - \pi/2) + 8\cos(\omega t - \pi/2) = -4\cos \omega t + 8 \sin \omega t = {}_\alpha\begin{pmatrix} 8 \\ -4 \end{pmatrix}$$

So $[h(x)]_\alpha = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$ And $[h]_\alpha^\alpha [x]_\alpha = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$

These are equal, as expected.

(c) Compute $[I]_\beta^\alpha$ and $[I]_\alpha^\beta$.

$$[w_1]_\alpha = \left[\sin \omega t \cos(-\pi/6) + \cos \omega t \sin(-\pi/6)\right]_\alpha = \left[\tfrac{\sqrt{3}}{2} \sin \omega t - \tfrac{1}{2} \cos \omega t\right]_\alpha = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}$$

$$[w_2]_\alpha = \left[\cos \omega t \cos(-\pi/6) - \sin \omega t \sin(-\pi/6)\right]_\alpha = \left[\tfrac{1}{2} \sin \omega t + \tfrac{\sqrt{3}}{2} \cos \omega t\right]_\alpha = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$$[I]_\beta^\alpha = \Big[\, [w_1]_\alpha \;\; [w_2]_\alpha \,\Big] = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \qquad [I]_\alpha^\beta = \left([I]_\beta^\alpha\right)^{-1} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

(d) Use the method from class for change of bases for a linear transformation, and the results of the previous parts of this problem, to compute $[h]_\beta^\beta$.

$$[h]_\beta^\beta = [I]_\alpha^\beta [h]_\alpha^\alpha [I]_\beta^\alpha = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}$$