EXAM 1

Math 216, 2013-2014 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number__________ TA____________________ Day/Time____________________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ____________________

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

Total Score __________ (/100 points)

1. *(18 pts)* The 3×3 matrix A can be row reduced to the identity matrix by the following sequence of elementary row operations, performed in the listed order:

Op. 1: 3 times the third row is added to the first row

Op. 2: the first row is multiplied by 2

Op. 3: 5 times the first row is added to the third row

(a) What are the elementary matrices E_1, E_2, E_3 that correspond to these operations?

$$E_1 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}$$

(b) Compute $\det A$, and A^{-1}, *without computing A itself. (Be sure to show your reasoning.)*

The row reduction of A can be written as $E_3 E_2 E_1 A = I$

So $A^{-1} = E_3 E_2 E_1$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 2 & 0 & 6 \\ 0 & 1 & 0 \\ 10 & 0 & 31 \end{pmatrix}}$$

And

$\det E_3 \det E_2 \det E_1 \det A = 1$

so

$1 \cdot 2 \cdot 1 \cdot \det A = 1$

$\boxed{\det A = 1/2}$

(c) The matrix B row reduces to its reduced row echelon form (below) by the exact same sequence of row operations.

$$\text{rref}(B) = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Without computing B itself, find a nontrivial relation between the columns ($\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$, $\vec{b}_4$) of B. *(Again, be sure to explain your reasoning.)*

The columns $\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4$ of rref(B) have the evident relation:

$$3\vec{c}_1 + 5\vec{c}_2 - \vec{c}_3 = \vec{0}$$

Row operations preserve relations between column vectors, so we thus also have

$$3\vec{b}_1 + 5\vec{b}_2 - \vec{b}_3 = \vec{0}$$

2. *(16 pts)* Consider the product R (below) of two upper triangular matrices.

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} d_1 & d_2 & d_3 \\ 0 & e_2 & e_3 \\ 0 & 0 & f_3 \end{pmatrix} = \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{pmatrix} = R$$

(a) What are the coefficients in the equation below giving $\vec{r}_2$ as a linear combination of the row vectors $\vec{d} = (d_1 \quad d_2 \quad d_3)$, $\vec{e} = (0 \quad e_2 \quad e_3)$, and $\vec{f} = (0 \quad 0 \quad f_3)$?

$$\vec{r}_2 = \underline{0}\,\vec{d} + \underline{b_2}\,\vec{e} + \underline{b_3}\,\vec{f}$$

(b) Argue from part (a) to conclude the first entry in $\vec{r}_2$.

$$\vec{r}_2 = b_2 (0 \quad e_2 \quad e_3) + b_3 (0 \quad 0 \quad f_3)$$

The first entries in these vectors are 0, thus the first entry of $\vec{r}_2$ is 0.

(c) Use an argument similar to that in parts (a) and (b) to conclude the first and second entries of $\vec{r}_3$.

$$\vec{r}_3 = 0\vec{d} + 0\vec{e} + c_3\vec{f}$$
$$= c_3 (0 \quad 0 \quad f_3)$$

These first two entries are zero, and thus likewise for $\vec{r}_3$.

(d) Find an expression giving the determinant of R.

$$\det R = \det\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{pmatrix} \det\begin{pmatrix} d_1 & d_2 & d_3 \\ 0 & e_2 & e_3 \\ 0 & 0 & f_3 \end{pmatrix}$$

$$= (a_1 b_2 c_3)(d_1 e_2 f_3)$$
$$= a_1 b_2 c_3 d_1 e_2 f_3$$

3. *(16 pts)* Use the permutation definition of determinant to compute the determinant of the matrix below.

$$M = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 2 & 4 \\ 3 & 1 & 5 \end{pmatrix}$$

σ	sgn(σ)	$a_{1\sigma(1)}\, a_{2\sigma(2)}\, a_{3\sigma(3)}$	summand
$1 \to 1,\ 2 \to 2,\ 3 \to 3$	(+)	$(1)(2)(5)$	10
$1 \to 1,\ 2 \to 3,\ 3 \to 2$	(−)	$(1)(4)(1)$	−4
$1 \to 2,\ 2 \to 1,\ 3 \to 3$	(−)	$(0)(4)(5)$	0
$1 \to 2,\ 2 \to 3,\ 3 \to 1$	(+)	$(0)(4)(3)$	0
$1 \to 3,\ 2 \to 1,\ 3 \to 2$	(+)	$(2)(4)(1)$	8
$1 \to 3,\ 2 \to 2,\ 3 \to 1$	(−)	$(2)(2)(3)$	−12

$$\boxed{\begin{array}{c} 2 \\ \parallel \\ \det(A) \end{array}}$$

4. *(17 pts)* Determine if $\{p_1, p_2, p_3\}$ is linearly independent or linearly dependent in the vector space of all polynomials.

$$p_1 = 3x^3 + x^2 + 3x \qquad p_2 = 2x^3 + 2x \qquad p_3 = x^3 - 2x^2 - 7x$$

We consider solutions to

$$c_1 p_1 + c_2 p_2 + c_3 p_3 = 0$$

$$(3c_1 + 2c_2 + 1c_3)x^3 + (1c_1 + 0c_2 - 2c_3)x^2 + (3c_1 + 2c_2 - 7c_3)x = 0$$

$$\begin{aligned} 3c_1 + 2c_2 + 1c_3 &= 0 \\ 1c_1 + 0c_2 - 2c_3 &= 0 \\ 3c_1 + 2c_2 - 7c_3 &= 0 \end{aligned}$$

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & -2 \\ 3 & 2 & -7 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Nontrivial solution would exist only if $\det = 0$; but

$$\det = (-)(2)\left(\det\begin{pmatrix} 1 & -2 \\ 3 & -7 \end{pmatrix}\right) + 0 + (-)(2)\left(\det\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\right)$$

$$= 2 + 0 + 14 = 16 \neq 0$$

So only the trivial solution exists, and thus the polynomials are independent.

5. *(17 pts)* Consider the set $V = \{f \in C^1 | f'(x) - x^2 f(x) = 0 \text{ for all } x\}$. Show that V is a vector space.

We will show V is a subspace of the known vector space C^1.

To show V is closed under addition, suppose f_1 and f_2 are in V, and thus that

$$f_1'(x) - x^2 f_1(x) = 0 \quad , \quad f_2'(x) - x^2 f_2(x) = 0$$

Then

$$\begin{aligned}
&(f_1+f_2)'(x) - x^2 (f_1+f_2)(x) \\
&= f_1'(x) + f_2'(x) - x^2 (f_1(x) + f_2(x)) \\
&= \left(f_1'(x) - x^2 f_1(x)\right) + \left(f_2'(x) - x^2 f_2(x)\right) \\
&= 0 + 0 = 0
\end{aligned}$$

Thus $f_1 + f_2$ is in V.

To show V is closed under scalar multiplication, suppose f is in V, so that

$$f'(x) - x^2 f(x) = 0$$

Then

$$\begin{aligned}
&(cf)'(x) - x^2 (cf)(x) \\
&= c f'(x) - c x^2 f(x) \\
&= c \left(f'(x) - x^2 f(x)\right) \\
&= c(0) = 0
\end{aligned}$$

Thus cf is in V.

6. *(16 pts)* The vectors $\vec{v}_1$ and $\vec{v}_2$ are in $\mathbb{R}^3$, and the pair $\{\vec{v}_1, \vec{v}_2\}$ is known to be independent.

(a) Show that this pair of vectors cannot span $\mathbb{R}^3$.

If the pair spanned V it would be a basis, contradicting the established fact that $\dim(\mathbb{R}^3) = 3$, since the pair includes only two vectors.

(b) Suppose that $\vec{v}_3$ is not a linear combination of $\vec{v}_1$ and $\vec{v}_2$. Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ must be a basis for $\mathbb{R}^3$.

This is three vectors in a three-dim space — so we need only show that the trio is independent.

Consider
$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

If c_3 were nonzero we could write $\vec{v}_3 = -\frac{c_1}{c_3}\vec{v}_1 - \frac{c_2}{c_3}\vec{v}_2$, contradicting our assumption about $\vec{v}_3$. So we must have $c_3 = 0$, and thus

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

$\{\vec{v}_1, \vec{v}_2\}$ is known to be independent, so we must have $c_1 = 0$ and $c_2 = 0$.

So the only solution is $c_1 = 0, c_2 = 0, c_3 = 0$, and thus the trio is independent and therefore a basis.