EXAM 3

Math 216, 2012-2013 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA________________ Day/Time________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. ________
2. ________
3. ________
4. ________
5. ________
6. ________

Total Score ________ (/100 points)

1. *(15 pts)* The matrices A and P are given below.

$$A = \begin{pmatrix} 14 & -4 & -16 \\ -3 & 3 & 4 \\ 9 & -3 & -10 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 3 & 2 & -4 \\ 1 & 2 & 1 \\ 2 & 1 & -3 \end{pmatrix}$$

(a) Show that the columns of P are eigenvectors of A.

$$A\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \checkmark$$

$$A\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \checkmark$$

$$A\begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \\ -9 \end{pmatrix} = 3\begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix} \checkmark$$

(b) Show that the columns of P form a basis for $\mathbb{R}^3$.

$$\det(P) = 3\det\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} - 1\det\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix} + 2\det\begin{pmatrix} 2 & -4 \\ 2 & 1 \end{pmatrix}$$

$$= -21 \quad + 2 \quad + 20 \quad = 1 \neq 0$$

So cols of P are independent, three in a 3-d space.
So they are a basis.

(c) Use the information from the previous two parts to compute $P^{-1}AP$ *without* multiplying the matrices directly. Explain your reasoning.

A basis of eigenvectors "diagonalizes" a matrix:

$$D = P^{-1}AP$$

And the diagonal entries are the eigenvalues. So

$$P^{-1}AP = D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

2. *(15 pts)* In this problem we consider the inner product space spanned by the functions $\{1, x, x^2\}$, using the L^2 inner product on the interval $[0, 1]$. Compute the angle in this inner product space between the vectors $f_1 = 1 + 2x$ and $f_2 = x^2$.

$$\|f_1\| = \sqrt{\langle f_1, f_1\rangle} = \sqrt{\int_0^1 (1+2x)^2\,dx} = \sqrt{\int_0^1 1+4x+4x^2\,dx}$$

$$= \sqrt{\left(x+2x^2+\tfrac{4}{3}x^3\right]_0^1} = \sqrt{\tfrac{13}{3}}$$

$$\|f_2\| = \sqrt{\langle f_2, f_2\rangle} = \sqrt{\int_0^1 (x^2)^2\,dx} = \sqrt{\int_0^1 x^4\,dx}$$

$$= \sqrt{\left(\tfrac{1}{5}x^5\right]_0^1} = \sqrt{1/5}$$

$$\langle f_1, f_2\rangle = \int_0^1 (1+2x)(x^2)\,dx = \int_0^1 x^2+2x^3\,dx$$

$$= \left(\tfrac{1}{3}x^3 + \tfrac{1}{2}x^4\right]_0^1 = \frac{5}{6}$$

$$\theta = \arccos\left(\frac{\langle f_1, f_2\rangle}{\|f_1\|\,\|f_2\|}\right) = \arccos\left(\frac{5/6}{\sqrt{13/3}\sqrt{1/5}}\right)$$

$$= \boxed{\arccos\left(\frac{5\sqrt{5}}{2\sqrt{39}}\right)}$$

3. *(15 pts)* In this problem we consider the following arithmetic relating to the matrix A.

$$A = \begin{pmatrix} 71/49 & 24/49 & 30/49 \\ 24/49 & 93/49 & 6/49 \\ 30/49 & 6/49 & 130/49 \end{pmatrix} = \begin{pmatrix} 2/7 & 3/7 & -6/7 \\ 6/7 & 2/7 & 3/7 \\ -3/7 & 6/7 & 2/7 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2/7 & 6/7 & -3/7 \\ 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \end{pmatrix}$$

(a) This equation represents a particular kind of diagonalizability of the matrix A. What kind of diagonalizability is it, what feature of which other matrix in this equation motivates this terminology, and what feature of A ensures that it will have this kind of diagonalizability?

A is orthogonally diagonalizable, because in the equation $A = PDP^{-1}$, P is orthogonal.

The fact that A is symmetric ensures that this will be possible

(b) Interpreting A as $[T]_{\mathcal{S}}^{\mathcal{S}}$, find (standard basis representations of vectors in) a basis $\mathcal{V}$ such that $[T]_{\mathcal{V}}^{\mathcal{V}}$ is diagonal.

$$A = P \quad D \quad P^{-1}$$

$$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{V}}^{\mathcal{S}} [T]_{\mathcal{V}}^{\mathcal{V}} [I]_{\mathcal{S}}^{\mathcal{V}}$$

The $\mathcal{V}$ that makes $[T]_{\mathcal{V}}^{\mathcal{V}} = D$ is the basis of columns of $P = [I]_{\mathcal{V}}^{\mathcal{S}}$. So,

$$\mathcal{V} = \left\{ \begin{pmatrix} 2/7 \\ 6/7 \\ -3/7 \end{pmatrix}, \begin{pmatrix} 3/7 \\ 2/7 \\ 6/7 \end{pmatrix}, \begin{pmatrix} -6/7 \\ 3/7 \\ 2/7 \end{pmatrix} \right\}$$

4. *(20 pts)* Find a fundamental set of solutions to the system

$$\begin{aligned} y_1' &= 3y_1 - 4y_2 \\ y_2' &= 4y_1 + 3y_2 \end{aligned} \qquad \vec{y}' = A\vec{y} \quad A = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$

$$p(\lambda) = \det\begin{pmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{pmatrix} = (3-\lambda)^2 + 16 = \lambda^2 - 6\lambda + 25$$

$$\lambda = \frac{6 \pm \sqrt{36 - 100}}{2} = 3 \pm 4i \qquad \lambda_1 = 3+4i,\ \lambda_2 = 3-4i$$

To find eigenvector $\vec{v}_1$ for λ_1:

$$A - \lambda_1 I = \begin{pmatrix} -4i & -4 \\ 4 & -4i \end{pmatrix}$$

$$\begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix} \begin{matrix} ②/4 \\ ① + i② \end{matrix} \Rightarrow x - iy = 0$$

$$x = iy$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} iy \\ y \end{pmatrix} = y\overbrace{\begin{pmatrix} i \\ 1 \end{pmatrix}}^{\vec{v}_1}$$

Complex solution: $e^{(3+4i)x}\begin{pmatrix} i \\ 1 \end{pmatrix}$

$$= e^{3x}(\cos 4x + i\sin 4x)\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} -e^{3x}\sin 4x \\ e^{3x}\cos 4x \end{pmatrix}}_{\text{also solutions}} + i \underbrace{\begin{pmatrix} e^{3x}\cos 4x \\ e^{3x}\sin 4x \end{pmatrix}}_{\text{also solutions}}$$

FSS: $\left\{ \begin{pmatrix} -e^{3x}\sin 4x \\ e^{3x}\cos 4x \end{pmatrix}, \begin{pmatrix} e^{3x}\cos 4x \\ e^{3x}\sin 4x \end{pmatrix} \right\}$

5. *(15 pts)* Suppose that $A = [T]_S^S$, and

$$[T]_{\mathcal{V}}^{\mathcal{V}} = \begin{pmatrix} \boxed{5} & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix} \quad \text{where} \quad \mathcal{V} = \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 5 \\ 6 \end{pmatrix}}_{\vec{v}_2}, \underbrace{\begin{pmatrix} 2 \\ 3 \\ 8 \\ 4 \end{pmatrix}}_{\vec{v}_3}, \underbrace{\begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v}_4} \right\}$$

Find a fundamental set of solutions to the system $\vec{y}' = A\vec{y}$.

Choosing the Jordan basis $\mathcal{V}$, a f.s.s. is

$$\{e^{xA}\vec{v}_1, e^{xA}\vec{v}_2, e^{xA}\vec{v}_3, e^{xA}\vec{v}_4\}$$

$$e^{xA}\vec{v}_1 = e^{5x}\vec{v}_1 = e^{5x}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}$$

$$e^{xA}\vec{v}_2 = e^{5x}\vec{v}_2 = e^{5x}\begin{pmatrix} 0 \\ 1 \\ 5 \\ 6 \end{pmatrix}$$

$$e^{xA}\vec{v}_3 = e^{5x}(\vec{v}_3 + x\vec{v}_2) = e^{5x}\begin{pmatrix} 2 \\ 3+x \\ 8+5x \\ 4+6x \end{pmatrix}$$

$$e^{xA}\vec{v}_4 = e^{5x}\left(\vec{v}_4 + x\vec{v}_3 + \frac{x^2}{2}\vec{v}_2\right) = e^{5x}\begin{pmatrix} -3+2x \\ -2+3x+\frac{x^2}{2} \\ 8x+\frac{5x^2}{2} \\ 1+4x+3x^2 \end{pmatrix}$$

6. *(20 pts)* The matrix A below has characteristic polynomial $p(\lambda) = (\lambda - 3)(\lambda - 2)^2$, and the two vectors $\vec{v}$ and $\vec{w}$ are known eigenvectors. Find a third vector $\vec{u}$ which, combined with $\vec{v}$ and $\vec{w}$, forms a Jordan basis for A.

$$A = \begin{pmatrix} 11 & -3 & -2 \\ 2 & 2 & 0 \\ 33 & -12 & -6 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$$

$$A\vec{v} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = 3\vec{v} \qquad A\vec{w} = \begin{pmatrix} 0 \\ 4 \\ -6 \end{pmatrix} = 2\vec{w}$$

A third Jordan basis vector u must correspond to the eigenvalue 2 because of the remaining multiplicity.

$$A - 2I = \begin{pmatrix} 9 & -3 & -2 \\ 2 & 0 & 0 \\ 33 & -12 & -8 \end{pmatrix}$$ has rank 2 and thus

the eigenvalue 2 has only the one eigenvector ...

So we must have

$$J = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = [T]_{\mathcal{V}}^{\mathcal{V}}, \quad \mathcal{V} = \{\vec{v}, \vec{w}, \vec{u}\}$$

and we solve for $\vec{u}$ with $(A - 2I)\vec{u} = \vec{w}$

$$\left(\begin{array}{ccc|c} 9 & -3 & -2 & 0 \\ 2 & 0 & 0 & 2 \\ 33 & -12 & -8 & -3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -3 & -2 & -9 \\ 0 & -12 & -8 & -36 \end{array}\right) \begin{array}{l} ②/2 \\ ① - \frac{9}{2}② \\ ③ - \frac{33}{2}② \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2/3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} ① \\ ②/-3 \\ ③ - 4② \end{array}$$

z free, choose $z=0$, then $x=1$, $y=3$.

So $\vec{u} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$