EXAM 2

Math 216, 2012-2013 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

Disc.: Number________ TA____________ Day/Time____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________

1. ________

2. ________

3. ________

4. ________

5. ________

6. ________

Total Score ________ (/100 points)

1. *([redacted] pts)* In this problem, we consider the five functions

$$\begin{aligned} f_1 &= 4\sin(x-\pi/2) \\ f_2 &= 3\sin(x-\pi/8) \\ f_3 &= 6\cos(x) - 3\sin(x-\pi/3) \\ f_4 &= 2\sin(x+\pi/6) \\ f_5 &= \cos(x) - \sin(x) \end{aligned}$$

Without using the Wronskain, decide if the collection $\{f_1, f_2, f_3, f_4, f_5\}$ is linearly independent or linearly dependent, and explain how you know.

By the angle addition formulas, all 5 of these functions are in span$\{\sin x, \cos x\}$, which is 2-dimensional.

Because $5 > 2$, these functions must be linearly dependent.

2. *([redacted] pts)* Find a fundamental set of solutions to the differential equation

$$y^{[5]} - 4y^{[4]} + 28y''' + 4y'' - 29y' = 0$$

(Hint: You may use the fact that $e^{2x}\cos(5x)$ is a solution.)

$\Rightarrow$ $2+5i,\ 2-5i$ are roots of $p(\lambda) = \lambda^5 - 4\lambda^4 + 28\lambda^3 + 4\lambda^2 - 29\lambda$

$$(\lambda-(2+5i))(\lambda-(2-5i)) = \lambda^2 - 4\lambda + 29$$

$$\begin{array}{r|l}
 & \lambda^3 \qquad\quad -\lambda \\
\hline
\lambda^2 - 4\lambda + 29 & \lambda^5 - 4\lambda^4 + 28\lambda^3 + 4\lambda^2 - 29\lambda \\
 & \underline{\lambda^5 - 4\lambda^4 + 29\lambda^3} \\
 & \qquad\qquad -\lambda^3 + 4\lambda^2 - 29\lambda \\
 & \qquad\qquad \underline{-\lambda^3 + 4\lambda^2 - 29\lambda} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}$$

$$\text{So } p(\lambda) = (\lambda-(2+5i))(\lambda-(2-5i))(\lambda^3-\lambda)$$
$$= (\lambda-(2+5i))(\lambda-(2-5i))(\lambda)(\lambda+1)(\lambda-1)$$

roots are $2+5i,\ 2-5i,\ 0,\ -1,\ 1$

yielding solutions (that are independent by a thm from class, and thus a fundamental set of solutions):

$$\{e^{2x}\cos 5x,\ e^{2x}\sin 5x,\ 1,\ e^{-x},\ e^{x}\}$$

3. ([redacted] *pts)* What is the form of a particular solution to the differential equation below? (Do not solve for the coefficients.)

$$y^{[4]} + 2y'' + y = xe^{2x}\sin(3x) + \cos(x) + 5$$

$$p(\lambda) = \lambda^4 + 2\lambda^2 + 1 = (\lambda^2+1)^2 = (\lambda+i)^2(\lambda-i)^2$$

- For "$xe^{2x}\sin 3x$", $a+bi = 2+3i$ is not a root of p.
  So we get terms: $(c_1x+c_0)e^{2x}\sin 3x + (d_1x+d_0)e^{2x}\cos 3x$
- For "$\cos x$", $a+bi = i$ <u>is</u> a root of p, with $m=2$.
  So we get terms: $Ax^2\cos x + Bx^2\sin x$
- For "5", $a+bi = 0$ is not a root of p.
  So we get the term: C

The form of the particular solution then is

$$y_p = (c_1x+c_0)e^{2x}\sin 3x + (d_1x+d_0)e^{2x}\cos 3x + Ax^2\cos x + Bx^2\sin x + C$$

4. ([redacted] *pts)* The image of the linear transformation $T : P_4 \to \mathbb{R}^7$ is a 3-dimensional subspace of $\mathbb{R}^7$ (Recall that P_4 is the vector space of all real polynomials of degree 4 or less). What is the dimension of the kernel of T? (Make sure to explain your reasoning!)

$\dim P_4 = 5$, because $\{1, x, x^2, x^3, x^4\}$ is a basis.

$$\dim(\ker(T)) + \dim(\operatorname{im}(T)) = \dim(\text{domain})$$

So

$$\dim(\ker(T)) + 3 = 5$$

and thus $\dim(\ker(T)) = 2$

5. *([redacted] pts)* In this problem you will be guided through a process that will result in finding a fundamental set of solutions for the equation below on the interval $(0, \infty)$.

$$L(y) = y'' + \left(3 - \frac{2}{x}\right) y' - \frac{6y}{x} = 0$$

Note that

$$\begin{aligned} L &= D^2 + \left(3 - \frac{2}{x}\right) D - \frac{6}{x} \\ &= \left(D - \frac{2}{x}\right)(D+3) \end{aligned}$$

(a) What is the dimension of the space of solutions to $L(y) = 0$?

This is a 2nd order equation, coefficients are continuous, $\beta_2 = 1 \neq 0$;

so dim $= 2$.

(b) Find the kernel of $(D+3)$ (that is, solve $y' + 3y = 0$), and explain how you know these functions must also solve $L(y) = 0$.

$p(\lambda) = \lambda + 3$, root is $r = -3$, so $y = C_1 e^{-3x}$ are the solutions.

$$L(y) = \left(\left(D - \tfrac{2}{x}\right)(D+3)\right)(y) = \left(D - \tfrac{2}{x}\right)\left((D+3)(y)\right)$$

$$= \left(D - \tfrac{2}{x}\right)(0) = 0$$

So these functions are also solutions to $L(y) = 0$.

(c) Confirm that x^2 is in the kernel of $\left(D - \frac{2}{x}\right)$.

$$\left(D - \tfrac{2}{x}\right)(x^2) = (x^2)' - \left(\tfrac{2}{x}\right)(x^2) = 2x - 2x = 0$$

(d) Your friend Bob argues that, since $\left(D - \frac{2}{x}\right)(x^2) = 0$, it also follows that

$$L(x^2) = \left(D - \frac{2}{x}\right)(D+3)(x^2) = (D+3)\left(D - \frac{2}{x}\right)(x^2) = (D+3)(0) = 0$$

Bob thus claims that x^2 is a solution to $L(y) = 0$. Is his reasoning valid? Explain.

No. $\left(D - \frac{2}{x}\right)$ and $(D+3)$ don't commute, because $\frac{2}{x}$ is not a constant.

(e) Find any solution to $(D+3)(y) = x^2$.

$y' + 3y = x^2$ Guess: $y = Ax^2 + Bx + C$

$(2Ax + B) + 3(Ax^2 + Bx + C) = x^2$

$(3A)x^2 + (2A + 3B)x + (B + 3C) = x^2$

$3A = 1 \Rightarrow A = 1/3$

$2A + 3B = 0 \Rightarrow B = -2/9$

$B + 3C = 0 \Rightarrow C = 2/27$

So $y = \frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}$ is a solution to the above equation.

(f) Show that the solutions to $(D+3)(y) = x^2$ are also solutions to $L(y) = 0$.

$$L(y) = \left(\left(D - \tfrac{2}{x}\right)(D+3)\right)(y) = \left(D - \tfrac{2}{x}\right)\left((D+3)(y)\right)$$

$$= \left(D - \tfrac{2}{x}\right)\underbrace{\left(x^2\right)}_{\text{from part (e)}} = \underbrace{0}_{\text{from part (c)}}$$

(g) Combine the previous results to form a fundamental set of solutions to the original equation.

e^{-3x} and $\frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}$ are independent in the 2-dim space of solutions, so

$$\left\{ e^{-3x},\ \tfrac{1}{3}x^2 - \tfrac{2}{9}x + \tfrac{2}{27} \right\}$$

is a fundamental set of solutions.

6. *([redacted] pts)* The functions y_1, y_2, y_3 are solutions to the linear differential equation

$$L(y) = q_3(x)y''' + q_2(x)y'' + q_1(x)y' + q_0(x)y = 0$$

where all of the q_i functions are known to be continuous, and q_3 is never zero. It is also known that

$$M = \begin{pmatrix} y_1(0) & y_2(0) & y_3(0) \\ y_1'(0) & y_2'(0) & y_3'(0) \\ y_1''(0) & y_2''(0) & y_3''(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

(a) Show that $u(x) = y_1 + y_2 + y_3$ is a solution to the equation $L(y) = 0$.

$$L(u) = L(y_1 + y_2 + y_3) = L(y_1) + L(y_2) + L(y_3)$$

$$= 0 + 0 + 0$$

$$= 0$$

(b) Find the initial values $u(0)$, $u'(0)$, $u''(0)$. *(Hint: Think about a vector that you might multiply by the matrix M.)*

$$u(0) = y_1(0) + y_2(0) + y_3(0)$$
$$u'(0) = y_1'(0) + y_2'(0) + y_3'(0)$$
$$u''(0) = y_1''(0) + y_2''(0) + y_3''(0)$$

Then

$$\begin{pmatrix} u(0) \\ u'(0) \\ u''(0) \end{pmatrix} = M \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(c) Use the initial values from part (b) to show that u must be the zero function.

The IVP : $L(y) = 0$, $y(0)=0, y'(0)=0, y''(0)=0$

is solved by $y=0$; and parts (a) and (b) above show that $y=u$ is also a solution.

The IVP satisfies the conditions of the existence/uniqueness theorem, so there is only one solution.

So we must have $u=0$.

(d) What does the result of the previous part tell you about the independence or dependence of $\{y_1, y_2, y_3\}$?

$u=0 \implies y_1+y_2+y_3 = 0$

This is a non-trivial linear combination of the functions $\{y_1, y_2, y_3\}$ that equals the zero function.

So $\{y_1, y_2, y_3\}$ is linearly dependent.